Seminar on K(K)-theory

Dedicated to the birth of Isabella Mesland

Leiden Noncommutative Geometry Seminar

Notes taken by Yuezhao Li
Mathematical Institute, Leiden University

2022 Spring Semester
Last modified: September 13, 2023

Acknowledgements

This manuscript serves as notes of the Leiden Noncommutative Geometry Seminar in 2022 Spring Semester. In the first half of the seminar we focused on the basic knowledge of KK-theory for C*-algebras, introducing the basic pictures of KK-theory due to Kasparov and Cuntz, and providing the basic properties of KK-theory. The main references for these are [3, 14]. A remarkable highlight is the talks on the Kasparov product using unbounded KK-theory (Bram Mesland, April 12 and 19). In the second half of the seminar we turned to more research-related fields, and to those topics of the speakers' own interest.

The seminar was organised by the Noncommutative Geometry research group of Leiden University. The idea to have such a seminar emerged during 2021 Fall Semester, among Jack Ekenstam, Yufan Ge, Yuezhao Li and their supervisors Bram Mesland and Francesca Arici.

I am grateful to my supervisor Bram Mesland, for offering me this nice chance to study KK-theory — a huge topic that I was interested in for a long time; and for allowing me to work as the principal organiser of this seminar. I also wish to thank all the participants: Prof. Dr. Bram Mesland, Prof. Dr. Francesca Arici, Jack Ekenstam, Yufan Ge, Dr. Dimitris Gerontogiannis (Leiden University); Prof. Dr. Marten Wortel (University of Pretoria); Dr. Rui Dong, Mick Gielen, Malte Leimbach, Georg Huppertz (Radboud University Nijmegen) and Stein Meerboer (Utrecht University). Without your active participation, the seminar could never become so successful.

Errors and typos are inevitable in this manuscript. Please use at your own risk. Comments are warmly welcome. You may find me at mailto:y.li@math.leidenuniv.nl.

We wish to dedicate the notes to Isa, who was born during the first weeks of our seminar.

Yuezhao Li (Leiden University)

Notations and symbols

Let A be a C*-algebra. We write $\mathbb{M}_n(A)$ for the C*-algebra of $(n \times n)$-matrices with entries in A. We write $\mathcal{M}(A)$ for multipliers of A, and $\mathcal{UM}(A)$ for unitary multipliers of A. We write $\mathcal{Q}(A)$ for the corona algebra $\mathcal{M}(A)/A$ which fits in the extension $A \rightarrowtail \mathcal{M}(A) \twoheadrightarrow \mathcal{Q}(A)$. By an extension of C*-algebras $A \rightarrowtail B \twoheadrightarrow C$ we shall mean a short exact sequence such that $A \subseteq B$ is a closed ideal, and C is isomorphic to the quotient C*-algebra B/A.

We write $\mathbb{I}A := \mathrm{C}([0,1], A)$ for the cylinder of A, $SA := \mathrm{C}_0((0,1), A)$ for the suspension of A and $CA := \mathrm{C}_0((0,1], A)$ for the cone of A.

Let E be a Hilbert A-module. We write $\mathbb{K}_A(E)$ for the C*-algebra of compact operators on E, and $\mathbb{B}_A(E)$ for the C*-algebra of bounded *adjointable* operators on E. In different literatures the authors may have different notations for $\mathbb{K}_A(E)$ (e.g. $\mathrm{End}^0_A(E)$) and for $\mathbb{B}_A(E)$ (e.g. $\mathrm{End}_A(E)$ or $\mathbb{L}_A(E)$). When E is merely a separable Hilbert space (=Hilbert $\mathbb{C}$-module), we will omit $\mathbb{C}$ and E by writing $\mathbb{K}$ and $\mathbb{B}$ for the C*-algebra of compact and bounded operators thereon.

The colour used in the colorboxes of this manuscript is `F4F0EC`. Can you guess why I choose it?

Contents

February 15, 2022

K-theory of C*-algebras I

Speaker: Jack Ekenstam (Leiden University)

A standard reference is [13, Chapter 4].

1.1 Definition of K_0

Let A be a unital C*-algebra.

Definition 1.1. An element $u \in A$ is *unitary* if $u^*u = 1 = uu^*$. A unitary *over* A is a unitary in $\mathbb{M}_n(A)$. Two unitaries $u, v \in A$ are *homotopic* if there is a continuous path of unitaries connecting them.

Example 1.2. If $u \in \mathbb{M}_n(\mathbb{C})$ is unitary. Then $u = vDv^*$ for some *unitary diagonal matrix* D. We use a "rotation" on each diagonal entry to connect D with 1_n. Then we obtain a homotopy $u \sim vDv^* \sim vv^* \sim 1_n$.

Definition 1.3. An element $p \in A$ is a *projection* if $p = p^* = p^2$. Two projections p, q are *unitarily equivalent* if there exists a unitary $u \in A$ such that $p = uqu^*$. Two projections $p, q \in A$ are *homotopic* if there is a continuous path of projections connecting them.

Example 1.4. If $p, q \in \mathbb{M}_n(\mathbb{C})$ are projections of the same rank k. Choose two basis such that p and q are $\begin{pmatrix} 1_k & \\ & 0_{n-k} \end{pmatrix}$ under the corresponding basis. Then a change of basis makes p and q unitarily equivalent.

Conversely: p and upu^* have the same rank.

Definition 1.5. $K_0(A)$ is the abelian group generated by homotopy classes of projections over A, with relations

1. $[0] = 0$.
2. $[p + q] = [p \oplus q]$.

Remark 1.6. Any $x \in K_0(A)$ can be written as $[p] - [q]$. And we can take $q = [1_k]$ using the following proposition:

Proposition 1.7. *If $p, q \in A$ are projections with $pq = 0 = qp$. Then $[p + q] = [p] + [q]$.*

Proof. We need to find a homotopy $p \oplus q \sim (p + q) \oplus 0$ in $\mathbb{M}_2(A)$. This is given by

$$\begin{pmatrix} p + q\cos^2 t & -q\sin t\cos t \\ q\sin t\cos t & q\sin^2 t \end{pmatrix}.$$

Now for the remark: take $x = [p] - [q] = [p \oplus (1 - q)] - [1]$. □

Proposition 1.8. *If $p, q \in A$ are projections with $\|p - q\| < 1$. Then there exists a unitary u such that $p = uqu^*$.*

Proof. Consider $x = qp + (1 - q)(1 - p)$. Then $xp = qp = qx$. One computes that

$$x - 1 = 2qp - q - p = 2qp - q^2 + q - p = (2q - 1)(p - q).$$

Notice that $2q - 1$ is a self-adjoint unitary, so has norm 1; $\|p - q\| < 1$ by assumption. Then

$$\|x - 1\| < \|2q - 1\|\|p - q\| < 1.$$

So x is invertible. Define $u = x(x^*x)^{-1/2}$. Some functional calculus shows that p commutes with $(x^*x)^{-1/2}$. Then

$$up = x(x^*x)^{-1/2}p = xp(x^*x)^{-1/2} = qx(x^*x)^{-1/2} = qu.$$ □

Corollary 1.9. *If $t \mapsto p_t$ is a path of projections. Then there exists a path of unitaries $t \mapsto u_t$ such that $p_t = u_t p_0 u_t^*$.*

Proof. By cutting the path into line segments which are small enough. □

Lemma 1.10. *If $u \in A$ is unitary. Then $u \oplus u^* \in \mathbb{M}_2(A)$ is also unitary and $u \oplus u^* \sim 1$.*

Proof. $t \mapsto \begin{pmatrix} u\cos t & -\sin t \\ \sin t & u^*\cos t \end{pmatrix}$ defines a homotopy $\begin{pmatrix} u & \\ & u^* \end{pmatrix} \sim \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Relacing u by 1 defines a homotopy $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \sim \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. □

Corollary 1.11. *If $p, q \in A$ are projections which are unitarily equivalent. Then $[p] = [q]$ in $\mathrm{K}_0(A)$.*

Proof.

$$[p] = \left[\begin{pmatrix} p & \\ & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} u & \\ & u^* \end{pmatrix}\begin{pmatrix} p & \\ & 0 \end{pmatrix}\begin{pmatrix} u^* & \\ & u \end{pmatrix}\right] = \left[\begin{pmatrix} q & \\ & 0 \end{pmatrix}\right] = [q].$$ □

Example 1.12. Let p and q be matrices over $\mathbb{C}$ of the same rank. Then they are unitarily equivalent, hence defines the same class in K_0.

Remark 1.13. K_0 is functorial: if $\phi\colon A \to B$ is a *-homomorphism. Then $\phi_*\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ by acting entrywise.

Now consider the case of non-unital C*-algebras.

Definition 1.14. Let A be non-unital. Let $\tilde{A}$ be the unitalisation of A. That is, $\tilde{A} = A \times \mathbb{C}$ with the product $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. A is a closed two-sided ideal of $\tilde{A}$: $A \to A \times \{0\} \subseteq \tilde{A}$. The quotient is $\tilde{A}/A \cong \mathbb{C}$. This induces a map $q_*\colon \mathrm{K}_0(\tilde{A}) \to \mathrm{K}_0(\mathbb{C})$. Define

$$\mathrm{K}_0(A) \coloneqq \ker q_*.$$

1.2 Definition of K_1

Let A be a unital C*-algebra.

Definition 1.15. $\mathrm{K}_1(A)$ is the abelian group generated by homotopy classes of unitaries over A with relations:

1. $[1] = 0$.
2. $[u] + [v] = [u \oplus v]$.

Proposition 1.16. *If $u, v \in A$ are unitaries. Then $[u] + [v] = [uv]$.*

Proof.

$$[uv] = [uv \oplus 1] = [(u \oplus 1)(v \oplus 1)] = [(u \oplus 1)(1 \oplus v)] = [u \oplus v].$$

Here we use a rotation to connect $v \oplus 1$ and $1 \oplus v$. □

Remark 1.17. For non-unital C*-algebras: $\mathrm{K}_1(A) = \ker(\mathrm{K}_1(\tilde{A} \to \mathrm{K}_1(\mathbb{C})))$. But as $\mathrm{K}_1(C) = 0$ (since every unitary is homotopic to the identity matrix). Then $\mathrm{K}_1(A) \cong \mathrm{K}_1(\tilde{A})$.

An alternative definition of K_1 is $\mathrm{K}_1(A) \coloneqq \mathrm{K}_0(SA)$ where $SA \coloneqq C_0((0,1), A)$.

1.3 Properties of K_0 and K_1

1.3.1 Homotopy invariance

Definition 1.18. A *homotopy* of *-homomorphisms $A \to B$ is a family of *-homomorphisms $\phi_t\colon A \to B$ such that for all $a \in A$: $t \mapsto \phi_t(a)$ is norm-continuous.

Two *-homomorphisms $\phi, \psi\colon A \rightrightarrows B$ are homotopic (denoted by $\phi \sim \psi$) if they are connected by a homotopy.

Remark 1.19. If $\phi, \psi\colon A \rightrightarrows B$ are homotopic. Then $\phi_* = \phi_*\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ and $\phi_* = \phi_*\colon \mathrm{K}_1(A) \to \mathrm{K}_1(B)$.

Definition 1.20. A *-homomorphism $\phi\colon A \to B$ is a *homotopy equivalence* if there exist a *-homomorphism $\psi\colon B \to A$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are both homotopic to the identity map.

By homotopy invariance: a homotopy equivalence induces an isomorphism in both K_0 and K_1.

1.3.2 Stability

Lemma 1.21. *For any $n \in \mathbb{N}$, $A \to \mathbb{M}_n(A)$ given by*

$$a \mapsto \begin{pmatrix} a & 0 & \\ 0 & 0 & \\ & & \ddots \end{pmatrix}$$

induces an isomorphism on K_0 and K_1.

Proof. Obviously it induces an isomorphism on K_0. For K_1: notice that

$$\mathrm{S}(\mathbb{M}_n(A)) = \mathrm{C}_0(0,1) \otimes \mathbb{M}_n \otimes A = \mathbb{M}_n(\mathrm{S}A).$$

So

$$\mathrm{K}_1(A) \cong \mathrm{K}_0(\mathrm{S}A) \cong \mathrm{K}_0(\mathbb{M}_n(\mathrm{S}A)) \cong \mathrm{K}_0(\mathrm{S}(\mathbb{M}_n(A))) = \mathrm{K}_1(\mathbb{M}_n(A)).$$ □

Proposition 1.22. *If $A_1 \subseteq A_2 \subseteq \cdots$ are C^*-subalgebras of A such that $\overline{\bigcup_n A_n} = A$. Then*

$$\varinjlim \mathrm{K}_0(A_i) \cong \mathrm{K}_0(A), \qquad \text{and} \qquad \varinjlim \mathrm{K}_1(A_i) \cong \mathrm{K}_1(A).$$

Example 1.23. We have $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$ induced by $p \mapsto \mathrm{rank}(p)$ and $\mathrm{K}_1(\mathbb{C}) = 0$ because every unitary matrix is homotopic to identity. Since $\mathbb{K} = \overline{\bigcup_n \mathbb{M}_n}$, we have $\mathrm{K}_0(\mathbb{K}) \cong \mathbb{Z}$ and $\mathrm{K}_1(\mathbb{K}) = 0$.

A similar statement shows that $\mathrm{K}_0(A) \cong \mathrm{K}_0(A \otimes \mathbb{K})$ and $\mathrm{K}_1(A) \cong \mathrm{K}_1(A \otimes \mathbb{K})$ for all A.

Definition 1.24. A and B are *stably isomorphic* if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

1.3.3 Long exact sequence

Let $J \overset{i}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} A/J$ be an extension of C^*-algebras. There is a 6-term long exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(J) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A/J) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A/J) & \longleftarrow & \mathrm{K}_1(A) & \longleftarrow & \mathrm{K}_1(J). \end{array}$$

Lemma 1.25.

$$\mathrm{K}_0(J) \xrightarrow{i_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J)$$

is exact in the middle.

Proof. By functoriality $\pi_* \circ i_* = 0$. This means $\mathrm{im}\, i_* \subseteq \ker \pi_*$. For the other way: if $x \in \ker \pi_*$ with $x = [p] - [1_n]$ for some p. Then $[\pi(p)] = [1_n]$, so $\pi(p)$ and 1_n are unitarily equivalent. Write

$$\begin{pmatrix} u & \\ & u^* \end{pmatrix} \begin{pmatrix} \pi(p) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & \\ & u \end{pmatrix} = 1.$$

A result in C^*-algebras shows that $\left(\begin{smallmatrix} u & \\ & u^* \end{smallmatrix}\right)$ lifts to a unitary in A. Then $vpv^* \equiv 1_n \mod J$. So $[p] \in \mathrm{im}\, i_*$.

For K_1: use that $A \mapsto \mathrm{S}A$ is exact. □

Definition 1.26. Let $\phi\colon A \to B$ be a *-homomorphism. The mapping cone of ϕ is the C^*-algebra

$$\{(a, f) \in A \times \mathrm{C}([0,1], B) \mid f(0) = 0,\ f(1) = \pi(a)\}.$$

Denote it by C_ϕ.

There is an extension

$$\mathrm{S}(A/J) \rightarrowtail \mathrm{C}_\pi \twoheadrightarrow A.$$

We will see that $\mathrm{K}_0(\mathrm{C}_\pi) \cong \mathrm{K}_0(J)$.

February 22, 2022

K-theory of C*-algebras II

Speaker: Yuezhao Li (Leiden University)

In this section, we establish a homological viewpoint of K-theory following [24, Chapter 11]: it is a homology theory of C*-algebras. This is in parallel with the standard algebraic topology of spaces, justifying the idea that C*-algebras are noncommutative spaces. Moreover, K-theory is universal among all stable homological functors. A somewhat surprising outcome is that K-theory, along with all such functors, automatically satisfies *Bott periodicity*. This was first observed by Cuntz in [8].

2.1 K-theory as a homology theory

What is K-theory? In the previous talk we have seen the definitions in detail. Now we turn to an algebraic topologist's viewpoint. K-theory is a homology theory of C*-algebras. Let C*Alg be the category of C*-algebras with *-homomorphisms as arrows, and Ab be the category of abelian groups.

Definition 2.1. A functor $F\colon \mathsf{C^*Alg} \to \mathsf{Ab}$ is called a *homological functor* if F is

- *Half-exact*: If $I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$ is an extension of C*-algebras, then $F(I) \xrightarrow{F(i)} F(E) \xrightarrow{F(q)} F(Q)$ is exact in the middle.
- *Homotopy invariant*: If $f, g\colon A \rightrightarrows B$ are homotopic then $F(f) = F(g)$.

K_0 is a homological functor. We want to turn it into a homology theory of C*-algebras.

Definition 2.2. A homology theory of C*-algebras is a collection of homotopy invariant functors $\{F_n\colon \mathsf{C^*Alg} \to \mathsf{Ab}\}_{n\in\mathbb{Z}}$ (or $\{F_n\colon \mathsf{C^*Alg} \to \mathsf{Ab}\}_{n\in\mathbb{N}}$) such that if

$$I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$$

is an extension of C*-algebras, then there is a collection of natural maps

$$\{\partial_n\colon F_n(Q) \to F_{n-1}(I)\}_{n\in\mathbb{Z}} \quad (\text{ or } \{\partial_n\colon F_n(Q) \to F_{n-1}(I)\}_{n\in\mathbb{N}}\ )$$

called *boundary maps*, such that the following sequence

$$\cdots \to F_n(I) \xrightarrow{F_n(i)} F_n(E) \xrightarrow{F_n(q)} F_n(Q) \xrightarrow{\partial_n} F_{n-1}(I) \to \cdots$$

is exact.

A homology theory is called half-infinite if $n \in \mathbb{N}$, and infinite if $n \in \mathbb{Z}$. Given a half-infinite homology theory, we do not know the exactness on one end. This does not happen for K-theory due to Bott periodicity. Our goal is to construct a (half-infinite) homology theory out of K_0, and extend to an infinite one using Bott periodicity.

Theorem 2.3. *Let* $F\colon \mathsf{C^*Alg} \to \mathsf{Ab}$ *be a homological functor. Define*

$$F_n := F \circ \mathrm{S}^n,$$

where S *is the suspension functor. Then* $\{F_n\colon \mathsf{C^*Alg} \to \mathsf{Ab}\}_{n\in\mathbb{N}}$ *is a homology theory.*

Sketch of the proof. Let $I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$ be an extension. The half-exactness of F and exactness of S imply that $F_n(I) \to F_n(E) \to F_n(Q)$ are exact in the middle. It suffices to construct the boundary maps. We only construct them but do not prove their exactness properties. Define $I \to \mathrm{C}_q$ by $x \mapsto (x, 0)$. One can show that it induces an isomorphism $F_0(I) \xrightarrow{\cong} F_0(\mathrm{C}_q)$ using both the homotopy-invariance and half-exactness of F.

There is an injective map $j\colon \mathrm{S}Q \rightarrowtail \mathrm{C}_q$ by sending $f \mapsto (0, f)$. (The quotient C_q/SQ is isomorphic to E). The boundary map is the composition

$$F_1(Q) = F_0(\mathrm{S}Q) \xrightarrow{F_0(j)} F_0(\mathrm{C}_q) \xrightarrow{\cong} F_0(I). \qquad \square$$

Since K_0 is a homological functor. The above construction applies to K-theory and yields a long exact sequence. Notice that our already defined K_1 fits in the long exact sequence as well.

Corollary 2.4. *Let* $F\colon \mathsf{C^*Alg} \to \mathsf{Ab}$ *be a homological functor. Then it is split-exact. That is, if*

$$I \overset{i}{\rightarrowtail} E \underset{s}{\overset{q}{\rightleftarrows}} Q$$

is a split extension. Then the sequence

$$F(I) \overset{F(i)}{\rightarrowtail} F(E) \overset{F(q)}{\twoheadrightarrow} F(Q)$$

is exact.

Proof. Since F is half-exact, the sequence is exact in the middle. It suffices to show that $F(i)$ is injective and $F(q)$ is surjective. $F(q)$ is surjective because $F(s)$ is a splitting. For the injectivity of $F(i)$, notice that $F_1(q)$ is also surjective where $F_1 := F \circ \mathrm{S}$. The long exact sequence therefore implies the boundary map $F_1(Q) \xrightarrow{\partial} F_0(I)$ is the zero map. Then the exactness at $F_0(I)$ implies $F(i)$ is injective. $\square$

2.2 Bott periodicity

Bott periodicity is the following theorem

Theorem 2.5 (Bott periodicity)**.** *There are natural isomorphisms* $\mathrm{K}_{n+2} \cong \mathrm{K}_n$ *for all* n*.*

As a consequence, an extension $I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$ of C^*-algebras gives an induced six-term cyclic exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \xrightarrow{i_*} & \mathrm{K}_0(E) & \xrightarrow{q_*} & \mathrm{K}_0(Q) \\ {\scriptstyle \mathrm{ind}}\uparrow & & & & \downarrow{\scriptstyle \exp} \\ \mathrm{K}_1(Q) & \xleftarrow[q_*]{} & \mathrm{K}_1(E) & \xleftarrow[i_*]{} & \mathrm{K}_1(I), \end{array}$$

where the boundary map $\mathrm{ind}\colon\ \mathrm{K}_1(Q) \to \mathrm{K}_0(I)$ is usually called the *index map*. The map $\exp\colon\ \mathrm{K}_0(Q) \to \mathrm{K}_1(I)$ is defined by the composition $\mathrm{K}_0(Q) \xrightarrow{\cong} \mathrm{K}_2(Q) \xrightarrow{\partial} \mathrm{K}_1(I)$, called the *exponential map*. In particular, we may define $\mathrm{K}_{-1} := \mathrm{K}_1$ to extend the half-infinite long exact sequence to an infinite one. ("The desuspension is the same as the suspension.")

Bott periodicity is essentially due to the $\mathbb{K}$-stability of K_0.

- $\mathbb{K}$-*stable*: Any corner embedding $e\colon A \hookrightarrow A \otimes \mathbb{K}$ induces an isomorphism $F(e)\colon F(A) \xrightarrow{\cong} F(A \otimes \mathbb{K})$.

Remark 2.6. Before proving the Bott periodicity, we first explain why we need $\mathbb{K}$-stability — or why it makes K-theory useful.

Let G be a compact group acting on a topological space X. We would like to understand the orbit space X/G, which is usually ill-behaved (e.g. non-Hausdorff), and $\mathrm{C}_0(X/G)$ becomes useless. In the philosophy of NCG, we should replace the algebra $\mathrm{C}_0(X/G)$ by a noncommutative one $\mathrm{C}_0(X) \rtimes G$, which is noncommutative but easier to describe. $\mathrm{C}_0(X/G)$ is Morita–Rieffel equivalent to $\mathrm{C}_0(X) \rtimes G$. So a reasonable homology theory for C^*-algebras should be the same for Morita–Rieffel equivalent C^*-algebras.

Any C^*-algebra A is Morita–Rieffel equivalent to $A \otimes \mathbb{K}$. Therefore such an embedding should be mapped to an isomorphism. This is $\mathbb{K}$-stability.

(One might also attempt to replace $\mathbb{K}$-stability by some weaker stability conditions like $\mathbb{M}_n$-stability. They yield some connective/unstable variants of K-theory.)

We shall prove that

Theorem 2.7 ([8]). *Let $F\colon \mathsf{C^*Alg} \to \mathsf{Ab}$ be a $\mathbb{K}$-stable homological functor, that is, F is homotopy invariant, half-exact and $\mathbb{K}$-stable. Then F satisfies Bott periodicity: that is, there is a natural isomorphism*

$$F(A) \cong F(\mathrm{S}^2 A).$$

The proof makes essential use of the Toeplitz algebra $\mathcal{T}$ and a closed ideal $\mathcal{T}_0 \subseteq \mathcal{T}$. A short review of the Toeplitz extension can be found in [1].

Definition 2.8. Let $S \in \mathbb{B}(\ell^2(\mathbb{N}))$ be the unilateral shift. That is,

$$S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots).$$

The Toeplitz algebra $\mathcal{T}$ is the C^*-subalgebra of $\mathbb{B}(\ell^2(\mathbb{N}))$ generated by S.

Exercise 2.9.
- $S^*S = 1$ and $SS^* = 1 - E_{00}$
- $\mathcal{T}$ contains all finite-rank operators, hence $\mathbb{K} \subseteq \mathcal{T}$ as a closed ideal.

Another characterisation is that the Toeplitz algebra is the universal C^*-algebra generated by an isometry S. That is, let A be a unital C^*-algebra. There is a bijection between {Isometries s in A} and {*-homomorphisms $\mathcal{T} \to A$ sending $S \mapsto s$}.

Definition 2.10. $\mathcal{T}_0$ is the closed ideal

$$\mathcal{T}_0 := \ker(\mathcal{T} \to \mathbb{C} \colon S \mapsto 1).$$

The quotient $\mathcal{T}/\mathbb{K}$ is isomorphic to $\mathrm{C}(\mathbb{T})$ (Why? Notice that S is essentially unitary, that is, its image is unitary in $\mathcal{T}/\mathbb{K}$; and has essential spectrum $\mathbb{T}$). The extension

$$\mathbb{K} \rightarrowtail \mathcal{T} \twoheadrightarrow \mathrm{C}(\mathbb{T})$$

is called the *Toeplitz extention.*

The map $\mathcal{T} \xrightarrow{S \mapsto 1} \mathbb{C}$ factors as $\mathcal{T} \twoheadrightarrow \mathrm{C}(\mathbb{T}) \xrightarrow{\mathrm{ev}_1} \mathbb{C}$. So the kernel $\mathcal{T}_0$ restricrts to $\mathcal{T}_0 \to \mathrm{C}_0(\mathbb{R})$ since $\mathrm{C}_0(\mathbb{R}) \cong \ker(\mathrm{C}(\mathbb{T}) \xrightarrow{\mathrm{ev}_1} \mathbb{C})$. This fits in the diagram

$$\begin{array}{ccccc} \mathbb{K} & \rightarrowtail & \mathcal{T}_0 & \twoheadrightarrow & \mathrm{C}_0(\mathbb{R}) \\ \| & & \downarrow & & \downarrow \\ \mathbb{K} & \rightarrowtail & \mathcal{T} & \twoheadrightarrow & \mathrm{C}(\mathbb{T}) \end{array}$$

Notice that $\mathrm{C}_0(\mathbb{R})$ is nuclear (because it is commutative). Then for *any* C^*-algebra A we have an extension

$$A \otimes_{\mathrm{sp}} \mathbb{K} \rightarrowtail A \otimes_{\mathrm{sp}} \mathcal{T}_0 \twoheadrightarrow A \otimes_{\mathrm{sp}} \mathrm{C}_0(\mathbb{R}) \tag{1}$$

where $\otimes_{\mathrm{sp}}$ denotes the *spatial* tensor product. But as $\mathrm{C}_0(\mathbb{R})$ and $\mathbb{K}$ are nuclear (hence also $\mathcal{T}_0$), these C^*-tensor products are unique and we may remove sp as well.

The extension (1) yields the long exact sequence:

$$\cdots F(\mathrm{S}(A \otimes \mathcal{T}_0)) \to F(\mathrm{S}(A \otimes \mathrm{C}_0(\mathbb{R}))) \xrightarrow{❀} F(A \otimes \mathbb{K}) \to F(A \otimes \mathcal{T}_0) \to \cdots$$

Notice that $\mathrm{S}(A \otimes \mathrm{C}_0(\mathbb{R})) = \mathrm{S}^2 A$. By $\mathbb{K}$-stability, $F(A \otimes \mathbb{K}) \cong F(A)$. To prove Bott periodicity, it suffices to show that the map ❀ is an isomorphism. This means $F(\mathrm{S}(A \otimes \mathcal{T}_0))$ and $F(A \otimes \mathcal{T}_0)$ are zero for any A.

Notice that if F is a $\mathbb{K}$-stable homological functor, then so are $F(\mathrm{S}(A \otimes -))$ and $F(A \otimes -)$. Therefore it suffices to prove that

Lemma 2.11. *If $F\colon \mathsf{C^*Alg} \to \mathsf{Ab}$ is a $\mathbb{K}$-stable homological functor. Then $F(\mathcal{T}_0) = 0$.*

How to prove the lemma? Intuitively, if $F(\mathcal{T}_0) = 0$, then given any split-exact sequence

$$\mathcal{T}_0 \xrightarrow{i} E \underset{s}{\overset{q}{\rightleftarrows}} Q,$$

we must have $F(q)$ and $F(s)$ are isomorphisms. Intuitively we may choose the obvious split extension

$$\mathcal{T}_0 \xrightarrow{i} \mathcal{T} \underset{s}{\overset{q}{\rightleftarrows}} \mathbb{C},$$

where $q\colon S \mapsto 1$ and s is the unique unital *-homormophism $\mathbb{C} \to \mathcal{T}$. Clearly $F(q) \circ F(s) = \mathrm{id}$ but proving $F(s) \circ F(q) = \mathrm{id}$ is usually difficult. $\mathbb{K}$-stability allows us to provide some extra space in order to construct a homotopy $s \circ q \sim \mathrm{id}$. Then the homotopy invariance of F implies $F(s) \circ F(q) = \mathrm{id}$. Set

$$\hat{\mathcal{T}} := \mathbb{K} \otimes \mathcal{T} + \mathcal{T} \otimes 1 \subseteq \mathcal{T} \otimes \mathcal{T}.$$

Then $\mathbb{K} \otimes \mathcal{T}_0$ is a closed ideal of $\hat{\mathcal{T}}$. In partular, $(\mathbb{K} \otimes \mathcal{T}_0) \cap (\mathcal{T} \otimes 1) = \emptyset$, so there is a split extension

$$\mathbb{K} \otimes \mathcal{T}_0 \xrightarrow{i} \hat{\mathcal{T}} \underset{s}{\overset{q}{\rightleftarrows}} \mathcal{T},$$

where s sends $\mathcal{T}$ to the copy $\mathcal{T} \otimes 1 \subseteq \hat{\mathcal{T}}$.

Proof of Lemma 2.11. We claim that $F(i)$ is the zero map. Then $F(\mathcal{T}_0) \cong F(\mathbb{K} \otimes \mathcal{T}_0) = 0$ since $F(i)$ is injective. For this we use the *additivity* of F:

- *Additivity*: If $f, g\colon A \rightrightarrows B$ are *orthogonal* *-homomorphisms between C*-algebras. Then $f + g$ is also a *-homomorphism and $F(f + g) = F(f) + F(g)$.

If F is a homological functor, then it is split-exact automatically additive. (Exercise).

Instead of i, we consider the *-homomorphism $j\colon \mathcal{T}_0 \hookrightarrow \mathbb{K} \otimes \mathcal{T}_0 \xrightarrow{i} \mathcal{T}$. Since F is $\mathbb{K}$-stable, $\mathcal{T} \hookrightarrow \mathbb{K} \otimes \mathcal{T}_0$ induces an isomorphism. Then it suffices to show that $F(j) = 0$.

Define the *-homomorphism

$$\phi_1\colon \mathcal{T}_0 \to \hat{\mathcal{T}}, \qquad x \mapsto (S \otimes 1)(x \otimes 1)(S^* \otimes 1) = \mathrm{Ad}_S\, x \otimes 1,$$

and set

$$\phi_0 := j + \phi_1\colon \mathcal{T}_0 \to \hat{\mathcal{T}}.$$

We claim that j and ϕ_1 are orthogonal and $F(\phi_0) = F(\phi_1)$. As a result

$$F(\phi_1) = F(\phi_0) = F(j + \phi_1) = F(j) + F(\phi_1) \quad \Longrightarrow \quad F(j) = 0 \quad \Longrightarrow \quad F(i) = 0.$$

It is easy to see that $j \perp \phi_1$ because $j(x) = E_{00} \otimes x = (1 - SS^*) \otimes x$ and $\phi_1(y) = SyS^* \otimes 1$. So $j(x) \cdot \phi_1(y) = \phi_1(y) \cdot j(x) = 0$. Therefore, $\phi_0 = j + \phi_1$ is a *-homomorphism.

Now we prove that $F(\phi_0) = F(\phi_1)$. Notice that $\mathcal{T}_0$ is generated by $S - 1$, and

$$\begin{aligned}
\phi_1(S-1) &= (S(S-1)S^*) \otimes 1 = S^2S^* \otimes 1 - SS^* \otimes 1 = S^2S^* \otimes 1 + E_{00} \otimes 1 - 1 \otimes 1 && =: S_1 - 1\\
\phi_0(S-1) &= S^2S^* \otimes 1 + E_{00} \otimes 1 - 1 \otimes 1 + E_{00} \otimes (S-1) = S^2S^* \otimes 1 + E_{00} \otimes S - 1 \otimes 1 && =: S_0 - 1.
\end{aligned}$$

So it suffices to find a path of isometries S_t in $\hat{\mathcal{T}}$. Then

$$\phi_t\colon \mathcal{T}_0 \to \hat{\mathcal{T}}, \quad S - 1 \mapsto S_t - 1$$

is a homotopy between ϕ_0 and ϕ_1. We can write $S_i = U_i \circ (S \times 1)$ for $i = 0, 1$, where

$$\begin{aligned}
U_0 &= S^2(S^*)^2 \otimes 1 + E_{00}S^* \otimes S + SE_{00} \otimes S^* + E_{00} \otimes E_{00},\\
U_1 &= S^2(S^*)^2 \otimes 1 + E_{00}S^* \otimes 1 + SE_{00} \otimes 1.
\end{aligned}$$

Both U_0 and U_1 are self-adjoint unitaries. For every self-adjoint unitary we may continuously move the -1 eigenvalue of it to $+1$ along the unit circle via functional calculus. Then U_i are both connected to 1 by a path of unitaries in $\hat{\mathcal{T}}$. Therefore S_0 and S_1 are connected by a path of isometries, and ϕ_0 is homotopic to ϕ_1. □

2.3 Thom isomorphism

A historical remark of K-theory:

- Grothendieck (1957): (algebraic) K-theory of varieties, working with coherent sheaves. "K" stands for *Klasse* ("class" in German).
- Atiyah, Hirzebruch (1959): (topological) K-theory of topological spaces, working with vector bundles. Roughly: "Classifying vector bundles over a topological space (up to stable equivalence)".
- Serre (1955), Swan (1962): vector bundles on compact spaces ~ projective modules over commutative rings (algebras).
- Many people[1] (1970s): topological K-theory for C*-algebras. Vector bundles are replaced by finitely-generated projective modules. Roughly: "Classifying vector bundles over a noncommutative space".

Let X be a locally compact Hausdorff space. The topological K-theory of X is the abelian group generated by formal differences of equivalence classes of vector bundles. There are isomorphisms

$$\mathrm{K}_n(\mathrm{C}_0(X)) \cong \mathrm{K}^{-n}(X).$$

Topological K-theory is a generalised cohomology theory of topological spaces. In particular, we have

Theorem 2.12 (Thom isomorphism)**.** *Let X be a compact Hausdorff space. Let E be an rank-k K-oriented real vector bundle over X. Then there are isomorphisms*

$$\mathrm{K}^n(E) \cong \mathrm{K}^{n+k}(X).$$

In the view of KK-theory, the Thom isomorphism has the following explanation. Let $E \to X$ be a real vector bundle. Then there is a KK-equivalence

$$\Gamma(\mathrm{C}\ell(E)) \sim_{\mathrm{KK}} \mathrm{C}_0(E),$$

where $\Gamma(\mathrm{C}\ell(E))$ is the C*-algebra of sections of the Clifford bundle of E.

If E is K-oriented, then the K-orientation gives a Morita equivalence (see Definition 3.20):

$$\Gamma(\mathrm{C}\ell(E)) \sim_{\mathrm{Morita}} \begin{cases} \mathrm{C}(X) & \text{if } \mathrm{rank}(E) \text{ is even,} \\ \mathrm{C}(X) \hat{\otimes} \mathbb{C}\ell_1 & \text{if } \mathrm{rank}(E) \text{ is odd.} \end{cases}$$

Here $\mathbb{C}\ell_1$ is the complex Clifford algebra generated by a single generator, equipped with the standard grading; $\hat{\otimes}$ denotes the graded tensor product. The graded Clifford algebra $\mathbb{C}\ell_1$ is KK-equivalent to $\mathrm{C}_0(\mathbb{R})$, hence yields only a dimension shift of the corresponding KK-theory group. Together with the previous KK-equivalence we have that $\mathrm{C}(X)$ is KK-equivalent to $\mathrm{C}_0(E)$ up to a dimension shift. We shall see later that KK-equivalent C*-algebras have the same K-theory.

The Connes–Thom isomorphism is an analog of the Thom isomorphism. For this one needs the conception of crossed product C*-algebras. Let G be a unimodular locally compact group and A be a G-C*-algebra, that is, a C*-algebra equipped with a G-action $G \xrightarrow{\alpha} \mathrm{Aut}(A)$. For $f, g \in \mathrm{C}_c(G, A)$, define

$$f^*(t) := \alpha_t(f(t^{-1}))^*,$$
$$f * g(t) := \int_G f(s)\alpha_s(g(s^{-1}t))\, \mathrm{d}\mu(s).$$

Then $\mathrm{C}_c(G, A)$ becomes a *-algebra. Its completion with respect to the L^1-norm is a Banach *-algebra, denoted by $L^1(G, A)$. The crossed product $A \rtimes_\alpha G$ is the completion of $L^1(G, A)$ with respect to a suitable C*-norm. (There are different choices of C*-norms in general, which yield different crossed product C*-algebras. If G is amenable, then all of them are the same.)

Example 2.13. If $G \xrightarrow{\alpha} \mathrm{Aut}(A)$ is the trivial action. Then $A \rtimes_\alpha G \cong A \otimes \mathrm{C}^*(G)$.

Theorem 2.14 (Connes)**.** *For any $\alpha \colon \mathbb{R}^n \to \mathrm{Aut}(A)$, $\mathrm{K}_{\bullet+n}(A \rtimes_\alpha \mathbb{R}^n) \cong \mathrm{K}_\bullet(A)$.*

If α is the trivial action, then $A \rtimes_\alpha \mathbb{R}^n \cong A \otimes \mathrm{C}_0(\mathbb{R})$ and the Connes–Thom isomorphism recovers the Bott periodicity.

[1] I do not know who exactly were among those pioneers that studied K-theory of C*-algebras. I am grateful if the reader can tell me.

2.4 Examples of K-theory groups

We have enough tools to compute the K-theory of some familiar C*-algebras: $\mathbb{C}$, $\mathbb{K}$, $\mathbb{B}$, $\mathcal{T}$, $C_0(\mathbb{R}^n)$, $C(\mathbb{T}^n)$.

Example 2.15. We have already seen that $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$. By $\mathbb{K}$-stability, $\mathbb{K}$ has the same K-theory with $\mathbb{C}$. In the proof of Bott periodicity we have seen that the K-theory of $\mathcal{T}$ agrees with the K-theory of $\mathbb{C}$. Exercise: write down a generator of $K_0(\mathcal{T})$.

Example 2.16. Let A be any C*-algebra. Define the *cone* of A as $CA := C_0((0,1], A) = C_0(0,1] \otimes A$. It is contractible, that is, the identity map $\mathrm{id}\colon CA \to CA$ is homotopic to the zero map. The homotopy is given by

$$F_t(f)(s) := f(ts).$$

Therefore, $K_0(CA) = K_1(CA) = 0$. Consider the extension

$$SA \rightarrowtail CA \overset{\mathrm{ev}_1}{\twoheadrightarrow} A.$$

It induces an long exact sequence

$$\cdots \to K_1(CA) \to K_1(A) \to K_0(SA) \to K_0(CA) \to \cdots .$$

Since CA has vanishing K-theory, we obtain $K_1(A) \cong K_0(SA)$ as desired.

Example 2.17. $K_0(\mathbb{B}) = K_1(\mathbb{B}) = 0$. $K_0(\mathbb{B}) = 0$ because, roughly speaking, we have projections and unitaries of infinite rank. To be precise: the isomorphism $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \cong \ell^2(\mathbb{N})$ induces a homotopy $\mathrm{id} \oplus \mathrm{id} \sim \mathrm{id}$, so $[\mathrm{id}] = 0$ in $K_0(\mathbb{B})$, forcing $K_0(\mathbb{B}) = 0$.

$K_1(\mathbb{B}) = 0$ because $K_1(A) = 0$ for any von Neumann algebra A. The logarithm function is a Borel function on $\mathbb{T}$. The path $t \mapsto \exp(it \operatorname{Log}(U))$ connects any unitary $U \in A$ to identity. So the unitary group of a von Neumann algebra is connected.

Example 2.18. The previous example can be generalised: let A be a C*-algebra. $\mathcal{M}(A \otimes \mathbb{K})$ is called the *stable multiplier algebra* of A. Then $K_0(\mathcal{M}(A \otimes \mathbb{K})) = K_1(\mathcal{M}(A \otimes \mathbb{K})) = 0$. $\mathbb{B}$ is the special case $A = \mathbb{C}$.

Example 2.19. By Bott periodicity: $K_0(C_0(\mathbb{R}^n)) \cong K_n(\mathbb{C})$ and $K_1(C_0(\mathbb{R}^n)) = K_{n+1}(\mathbb{C})$.

Example 2.20. Identify $\mathbb{T}$ with the one-point compactification of $(0,1) \cong \mathbb{R}$. The evaluation map ev_1 induces a split extension

$$C_0(\mathbb{R}) \rightarrowtail C(\mathbb{T}) \underset{\curvearrowleft}{\overset{\mathrm{ev}_1}{\twoheadrightarrow}} \mathbb{C}$$

By split-exactness of K-theory: $K_\bullet(C(\mathbb{T})) = K_\bullet(\mathbb{C}) \oplus K_\bullet(C_0(\mathbb{R})) = K_\bullet(\mathbb{C}) \oplus K_{\bullet+1}(\mathbb{C}) = \mathbb{Z}$.

Inductively: consider $\mathbb{T}^n \cong \mathbb{T}^{n-1} \times \mathbb{T}$ and the evaluation map for the last entry. We obtain a similar split extension

$$C_0(\mathbb{T}^{n-1} \times \mathbb{R}) \rightarrowtail C(\mathbb{T}^n) \underset{\curvearrowleft}{\overset{\mathrm{ev}_1}{\twoheadrightarrow}} C(\mathbb{T}^{n-1})$$

So $K_\bullet(C(\mathbb{T}^n)) = K_\bullet(C(\mathbb{T}^{n-1})) \oplus K_\bullet(C_0(\mathbb{T}^{n-1} \times \mathbb{R})) = K_\bullet(C(\mathbb{T}^{n-1})) \oplus K_{\bullet+1}(C_0(\mathbb{T}^{n-1})) = \mathbb{Z}^{2^{n-1}}$.

Exercise: write down the generators of $K_\bullet(C(\mathbb{T}^n))$.

March 1 and March 8, 2022

Hilbert C*-modules

Speaker: Jack Ekenstam (Leiden University)

The main reference for this section is [17].

3.1 Inner-product modules and Hilbert C*-modules

Definition 3.1. Let A be a C*-algebra. An *inner-product A-module* is a $\mathbb{C}$-linear space E that is also a right A-module, satisfying

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \qquad \text{for all } \lambda \in \mathbb{C},\ a \in A \text{ and } x \in E,$$

together with a linear map $\langle\cdot,\cdot\rangle\colon E \times E \to A$ such that:

1. $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$.
2. $\langle x, ya\rangle = \langle x, y\rangle a$.
3. $\langle y, x\rangle = \langle x, y\rangle^*$.
4. $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$.

We call E a *semi-inner-product module* if 4 is replaced by

4'. $\langle x, x\rangle \geq 0$.

Proposition 3.2. *If E is a semi-inner-product A-module, then $\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle$.*

Proof. WLOG $\|\langle x, x\rangle\| = 1$. For any $a \in A$:

$$\begin{aligned} 0 \leq \langle xa - y, xa - y\rangle &= a^*\langle x, x\rangle a - \langle y, x\rangle a - a^*\langle y, y\rangle + \langle y, y\rangle \\ &\leq a^*a - \langle y, x\rangle a - a^*\langle y, y\rangle + \langle y, y\rangle \\ &\leq a^*a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle. \end{aligned}$$

Choose $a = \langle x, y\rangle$. Then $a^*a \leq \langle y, y\rangle$. □

Let E be an inner-product module. Let $x \in E$. Define $\|x\| := \|\langle x, x\rangle\|^{1/2}$. Then $\|\langle x, y\rangle\| \leq \|x\|\|y\|$. So $\|\cdot\|$ is a norm on E.

If E is just an semi-inner-product module, then $\|\cdot\|$ is a semi-norm on E. Define $N := \{x \in E \mid \langle x, x\rangle = 0\}$. Then E/N is an inner-product module by setting

$$\langle x + N, y + N\rangle_{E/N} := \langle x, y\rangle_E.$$

E is, in particular, a normed A-module:

$$\|xa\|^2 = \|\langle xa, xa\rangle\| = \|a^*\langle x, x\rangle a\| \leq \|x\|^2\|a^*a\| = (\|x\|\|a\|)^2.$$

Definition 3.3. A *Hilbert A-module* is a complete inner-product A-module.

3.1.1 Examples of Hilbert C*-modules

Example 3.4.

- Let A be a C*-algebra, and E_0 be an inner-product A-module. Then the completion $E := \overline{E_0}$ is a Hilbert A-module.
- Let A_0 be a pre-C*-algebra. We can define inner-product A_0-modules in a similar fashion. Let E_0 be an inner-product A_0-module. Let $A := \overline{A_0}$ and $E := \overline{E_0}$. Then E is a Hilbert A-module.

Example 3.5. Let A be a C*-algebra and E be a Hilbert A-module. Let (e_i) be an approximate unit for A. For $x \in E$, $\langle xe_i - x, xe_i - x\rangle \to 0$. So EA is dense in E (and $x1 = x$ if A is unital). If A is not unital, then E is also a Hilbert A^+-module in a natural way.

Example 3.6. Let A be a C*-algebra and E be a Hilbert A-module. Define

$$B := \langle E, E\rangle = \{\langle x, y\rangle \mid x, y \in E\}.$$

This is a closed ideal in A. EB is dense in E by the previous calculation.

B need not be the whole of A. If $B = A$, we say A is *full*.

A non-full example: let $E = A = \mathrm{C}(\mathbb{T})$ and $F = \mathrm{C}_0(\mathbb{T} \setminus \{1\})$. Then $\langle F, F\rangle \neq \mathrm{C}(\mathbb{T})$.

Example 3.7. Let A be a C^*-algebra. Then $E = A$ is a Hilbert A-module with $\langle a, b\rangle := a^*b$. More generally: let $J \subseteq A$ be a closed right ideal. Then J is a Hilbert A-submodule of A.

Example 3.8. Let $\{E_i\}_{i\in I}$ be a set of Hilbert A-modules. Define

$$\bigoplus_{i\in I} E_i := \left\{ (x_i)_{i\in I} \mid x_i \in E_i,\ \sum_{i\in I} \langle x_i, x_i\rangle \text{ converges} \right\}.$$

This is a Hilbert A-module by setting $\langle (x_i), (y_i)\rangle := \sum_{i\in I}\langle x_i, y_i\rangle$.

Example 3.9. Let A be a C^*-algebra. Let $\mathcal{H}$ be a Hilbert space with basis $\{\xi_i\}_i \in I$. Then $\mathcal{H} \otimes_{\mathrm{alg}} A$ is an inner-product A-module. Its completion $\mathcal{H} \otimes A$ is a Hilbert A-module. If $\mathcal{H}$ is infinite-dimensional, we write $\mathcal{H}_A := \mathcal{H} \otimes A$. In general: $\mathcal{H} \otimes A \cong \oplus_{i\in I} A$.

Example 3.10 (Localisation of Hilbert C^*-modules). Let A be a unital C^*-algebras and $B \subseteq A$ be a unital C^*-subalgebra. A *conditional expectation* from A to B is a linear contractive idempotent $\psi\colon A \to B$. A conditional expectation is always positive and satisfies

$$\psi(bab') = b\psi(a)b', \quad \text{for } a \in A \text{ and } b, b' \in B.$$

We say ψ is *faithful*, if

$$a \geq 0,\ \psi(a) = 0 \quad \Longrightarrow \quad a = 0.$$

If E is a Hilbert A-module, then $\langle x, y\rangle_B := \psi(\langle x, y\rangle_A)$ defines a *semi-inner-product* B-module structure for E. It is an *inner-product* if ψ is faithful. In particular: A is a Hilbert B-module with $\langle x, y\rangle_B := \psi(\langle x, y\rangle_A)$.

Let E be a Hilbert A-module and $F \subseteq E$ be a Hilbert A-submodule. Then

$$F^\perp := \{y \in E \mid \langle x, y\rangle = 0 \text{ for all } x \in F\}$$

is a Hilbert A-submodule of E. **Warning**. In general $E \neq F \oplus F^\perp$.

Example 3.11. Let $E = A = \mathrm{C}(\mathbb{T})$ and $F = \mathrm{C}_0(\mathbb{T} \setminus \{1\})$. Then $F^\perp = \{0\}$.

3.2 Adjointable operators

Definition 3.12. Let E and F be Hilbert A-modules. The *adjointable operators* from E to F is the set

$$\mathbb{B}_A(E, F) := \{T\colon E \to F \mid \text{There exists } T^*\colon F \to E \text{ such that } \langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x \in E \text{ and } y \in F\}.$$

Thus $T \in \mathbb{B}_A(E, F)$ is automatically A-linear.

Let $T \in \mathbb{B}_A(E, F)$. Take $x \in E$ with $\|x\| \leq 1$. Set

$$T_x\colon F \to A, \qquad T_x(y) := \langle Tx, y\rangle.$$

Then $\|T_x(y)\| \leq \|T^*y\|$. So $\{\|T_x\| \mid \|x\| \leq 1\}$ is bounded by Banach–Steinhaus. Therefore T is bounded.

But not every bounded operator is adjointable! Consider

Example 3.13. Let $F = A = \mathrm{C}(\mathbb{T})$, and $E = \mathrm{C}_0(\mathbb{T} \setminus \{1\})$. Consider the inclusion $i\colon E \to F$. If i is adjointable, then i^* must satisfy $i^*(1) = 1$, but $1 \notin E$.

Some calculi for adjointable operators:

- If $T \in \mathbb{B}_A(E, F)$, then $T^* \in \mathbb{B}_A(E, F)$.
- If $T \in \mathbb{B}_A(E, F)$ and $S \in \mathbb{B}_A(F, G)$. Then $S \circ T \in \mathbb{B}_A(E, G)$.
- $\mathbb{B}_A(E) := \mathbb{B}_A(E, E)$ is a C^*-algebra: notice that

$$\|T\|^2 = \sup_{\|x\|\leq 1} \|\langle Tx, Tx\rangle\|^2 = \sup_{\|x\|\leq 1} \|\langle T^*Tx, x\rangle\| \leq \|T^*T\|.$$

For the other direction use Cauchy–Schwarz.

Proposition 3.14. *Let $T \in \mathbb{B}_A(E,F)$ and $x \in E$. Then $|Tx| \leq \|T\||x|$, where $|x| := \langle x, x\rangle^{1/2}$.*

Proof. Let ρ be any state of A. Then $\rho(\langle\cdot,\cdot\rangle)$ is a semi-inner-product. We iteratively use Cauchy–Schwartz:

$$\begin{aligned}
\rho(|Tx|^2) = \rho(\langle Tx, Tx\rangle) = \rho(\langle T^*Tx, x\rangle) &\leq \rho(\langle T^*Tx, T^*Tx\rangle)^{\frac{1}{2}}\rho(\langle x, x\rangle)^{\frac{1}{2}} \\
&= \rho(\langle (T^*T)^2x, x\rangle)^{\frac{1}{2}}\rho(\langle x, x\rangle)^{\frac{1}{2}} \\
&\leq \rho(\langle (T^*T)^2x, (T^*T)^2x\rangle)^{\frac{1}{4}}\rho(\langle x, x\rangle)^{\frac{1}{2}+\frac{1}{4}} \\
&\leq \cdots \\
&\leq \rho(\langle (T^*T)^{2^n}x, x\rangle)^{\frac{1}{2^n}}\rho(\langle x, x\rangle)^{1-\frac{1}{2^n}} \\
&\leq \|T^*T\|(\|x\|^2)^{\frac{1}{2^n}}\rho(|x|^2)^{1-\frac{1}{2^n}}
\end{aligned}$$

Let $n \to \infty$. Then $\rho(|Tx|^2) \leq \|T\|^2\rho(|x|^2) = \rho(\|T\|^2|x|^2)$. This holds for all states ρ of A, hence

$$|Tx|^2 \leq \|T\|^2|x|^2.$$

Taking the square root we obtain $|Tx| \leq \|T\||x|$. □

3.3 Compact operators

Let E and F be Hilbert A-modules. Let $x \in E$ and $y \in F$. Define the operator

$$\Theta_{x,y}\colon F \to E, \qquad \Theta_{x,y}(z) := x\langle y, z\rangle.$$

Then Θ is adjointable and $\Theta^*_{x,y} = \Theta_{y,x}$.

Definition 3.15. The compact operators from F to E is the set

$$\mathbb{K}_A(F,E) := \overline{\operatorname{span}}\{\Theta_{x,y} \mid x \in E, y \in F\}.$$

We have:

- $\Theta_{x,y}\Theta_{u,v} = \Theta_{x\langle y,u\rangle,v}$.
- $\Theta_{tx,y} = t\Theta_{x,y}$.
- $\Theta_{x,y}S = \Theta_{x,S^*y}$ for S adjointable.

Therefore $\mathbb{K}_A(E)$ is an ideal of $\mathbb{B}_A(E)$.

Example 3.16.
- Let $E = A$. Then $\mathbb{K}_A(A) \cong A$: $\Theta_{a,b} \mapsto ab^*$. This is clear if A is unital. If A is non-unital, then we may either use the fact that $\{ab \mid a, b \in A\}$ is dense in A by the existence of an approximate unit; or use Cohen–Hewitt factorization theorem.
- If A is unital, then $\mathbb{K}_A(A) = \mathbb{B}_A(A)$ because every adjointable operator T satisfies

  $$\langle Tx, y\rangle = (Tx)^*y = x^*(T1)^*y = \langle T(1)x, y\rangle,$$

  So $Tx = T(1)x = \Theta_{T(1),1}(x)$.
- $\mathbb{K}_A(E^m, F^n) \cong \mathbb{M}_{m\times n}(\mathbb{K}_A(E,F))$ and $\mathbb{B}_A(E^m, F^n) \cong \mathbb{M}_{m\times n}(\mathbb{B}_A(E,F))$.

Definition 3.17 (Strict topology)**.** The strict topology on $\mathbb{B}_A(E,F)$ is given by the semi-norms

$$T \mapsto \|Tx\|, \quad T \mapsto \|T^*x\|, \quad x \in E.$$

Proposition 3.18 ([17, Proposition 1.3])**.** *$\mathbb{K}_A(E,F)$ is strictly dense in the unit ball of $\mathbb{B}_A(E,F)$.*

Theorem 3.19. *Every $T \in \mathbb{K}_A(E,A)$ is given by $Ty = \langle x, y\rangle$ for some $x \in E$.*

3.3.1 Morita equivalence

Definition 3.20 (Morita equivalence)**.** Let A and B be C^*-algebras. We say they are *(strongly) Morita equivalent*, denoted by $A \sim_{\text{Morita}} B$, if there exists a *full* Hilbert A-module E such that $B \cong \mathbb{K}_A(E)$.

Theorem 3.21. *If A and B are σ-unital C^*-algebras. Then $A \sim_{\text{Morita}} B$ iff $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.*

3.4 Operations on Hilbert C*-modules

3.4.1 Exterior tensor product

Let E be a Hilbert B-module and F be a Hilbert C-module. Then the algebraic tensor product $E \otimes_{\text{alg}} F$ is naturally a right $B \otimes_{\text{alg}} C$-module:

$$(e \otimes f) \cdot (b \otimes c) := eb \otimes fc.$$

Notice that $B \otimes_{\text{alg}} C$ is dense in $B \otimes_{\text{sp}} C$. And we define

$$\langle e \otimes f, e' \otimes f' \rangle_{E \otimes F} := \langle e, e' \rangle_E \otimes \langle f, f' \rangle_F.$$

This is a semi-inner-product. We may quotient it by the elements

$$N := \{x \in E \otimes_{\text{alg}} F \mid \langle x, x \rangle_{E \otimes F} = 0\}$$

to obtain a inner-product module, and complete to a Hilbert $B \otimes_{\text{sp}} C$-module.

Remark 3.22. According to the discussion on [17, Page 34], the semi-inner-product defined above is indeed an *inner-product*. That is, $N = 0$. The proof is based on Kasparov's stablisation theorem (Theorem 3.28). See [17, Page 62].

Definition 3.23. The exterior tensor product of E and F is the Hilbert $B \otimes_{\text{sp}} C$-module

$$E \otimes F := \overline{E \otimes_{\text{alg}} F/N}^{\langle \cdot, \cdot \rangle_{E \otimes F}}.$$

3.4.2 Interior tensor product

Let E be a Hilbert B-module and F be a Hilbert A-module. Let $\phi \colon B \longrightarrow \mathbb{B}_A(F)$ be a *-homomorphism. Then B acts on F (on the left): $b \cdot f := \phi(b)f$. Define the A-valued semi-inner-product on $E \otimes_{\text{alg}} F$:

$$\langle e \otimes f, e' \otimes f' \rangle_{E \otimes_\phi F} := \langle f, \phi(\langle e, e' \rangle_E f') \rangle_F.$$

Definition 3.24. The interior tensor product of E and F is the Hilbert A-module

$$E \otimes_\phi F = \overline{E \otimes_{\text{alg}} F/N}^{\langle \cdot, \cdot \rangle_{E \otimes_\phi F}},$$

where

$$N = \{x \in E \otimes_{\text{alg}} F \mid \langle x, x \rangle_{E \otimes_\phi F} = 0\}.$$

One can check that N is generated by the elements of the form

$$(e \cdot b) \otimes f - e \otimes (b \cdot f).$$

Remark 3.25. How to check the positivity argument? We have

$$\begin{aligned}\langle e \otimes f, e' \otimes f' \rangle_{E \otimes_\phi F} &= \langle f, \phi(\langle e, e \rangle_E) f \rangle_F \\ &= \langle (\phi(\langle e, e \rangle_E))^{1/2} f, (\phi(\langle e, e \rangle_E))^{1/2} f \rangle_F \geq 0.\end{aligned}$$

Here we have used $\phi(\langle e, e \rangle_E) \geq 0$ because $\langle e, e \rangle_E \geq 0$ and ϕ is a *-homomorphism.

3.4.3 Pushout

Let E be a Hilbert B-module. Let $f\colon B \twoheadrightarrow A$ be a surjective *-homomorphism. Define

$$N_f := \{x \in E \mid f(\langle x, x\rangle_E) = 0\}.$$

This is an ideal in E. Then E/N_f is a right A-module with

$$q(x)f(b) := q(xb)$$

where $q\colon E \twoheadrightarrow E/N_f$ is the quotient map. It is easy to check this does not depend on the choice of b.

Define the A-valued inner-product on E/N_f:

$$\langle q(x), q(y)\rangle_{E_f} := f(\langle x, y\rangle_E).$$

Definition 3.26. The pushout of E along f is the Hilbert A-module

$$E_f := \overline{E/N_f}^{\langle\cdot,\cdot\rangle_{E_f}}$$

In particular:

Lemma 3.27. $E_f \cong E \otimes_f A$.

3.5 Kasparov's stablisation theorem

Theorem 3.28 (Kasparov's stablisation theorem)**.** *Let E be a countably-generated Hilbert B-module. Then $E \oplus \mathcal{H}_B \cong \mathcal{H}_B$, where $\mathcal{H}_B := \mathcal{H} \otimes B$.*

Corollary 3.29. *Any countably-generated Hilbert B-module is of the form $E \cong P\mathcal{H}_B$ for some bounded projection $P \in \mathbb{B}_B(\mathcal{H}_B)$.*

Corollary 3.30. *E is countably-generated iff $\mathbb{K}(E)$ is σ-unital.*

Remark 3.31. Let E be a Hilbert B-module. A (tight, normalised) *frame* of E is a set of elements $\{x_i\}_{i\in I}$ of E, such that for every $e \in E$ one has

$$\sum_{i\in I} x_i\langle x_i, e\rangle = e.$$

Note that x_i's are not necessarily orthogonal to each other. (See [11] for general frames in Hilbert C*-modules).

From Kasparov's stablisation theorem we have

Theorem 3.32. *Every countably-generated Hilbert* C**-module has a frame.*

Proof. We first apply the unitalisation to obtain $E \oplus \mathcal{H}_{B^+} \cong \mathcal{H}_{B^+}$. Kasparov's stablisation theorem implies that there is an isometry $V\colon E \to \mathcal{H}_{B^+}$. $\mathcal{H}_{B^+}$ has a basis $\{e_i \otimes 1\}_{i\in\mathbb{N}}$. Define

$$x_i := V^*(e_i \otimes 1).$$

Then $\{x_i\}_{i\in\mathbb{N}}$ is a frame of E:

$$\begin{aligned}
\sum_{i\in\mathbb{N}} x_i\langle x_i, e\rangle &= \sum_{i\in\mathbb{N}} V^*(e_i\otimes 1)\langle V^*(e_i\otimes 1), e\rangle \\
&= \sum_{i\in\mathbb{N}} V^*(e_i\otimes 1)\langle (e_i\otimes 1), Ve\rangle \\
&= V^*\left(\sum_{i\in\mathbb{N}} (e_i\otimes 1)\langle (e_i\otimes 1), Ve\rangle\right) \\
&= V^*Ve = e.
\end{aligned}$$

Note that $\{x_i\}_{i\in\mathbb{N}}$ is not a basis because $VV^* \neq 1$, so they are not orthogonal to each other. □

March 8 and March 22, 2022

KK-theory: Kasparov's picture

Speaker: Yufan Ge (Leiden University)

In this section, all C*-algebras are *separable*, and all Hilbert C*-modules are *countably-generated.*

4.1 Definition of Kasparov modules

Definition 4.1. A *graded* C**-algebra* is a C*-algebra A together with an automorphism $\beta_A\colon A \to A$ satisfying $\beta_A^2 = \mathrm{id}$.

Then A decomposes as a direct sum of Banach spaces $A = A_0 \oplus A_1$, where A_i is the $(-1)^i$ eigenspace of β. The decomposion is given by

$$a = \frac{a + \beta_A(a)}{2} + \frac{a - \beta_A(a)}{2}.$$

We say $\deg(a) = 0$ if $a \in A_0$, and $\deg(a) = 1$ if $a \in A_1$. We say a is *homogeneous* if $a \in A_0$ or $a \in A_1$.

Definition 4.2. Let (A, β_A) and (B, β_B) be graded C*-algebras. A *graded* **-homomorphism* is a *-homomorphism $\phi\colon A \to B$ such that the diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ {\scriptstyle \beta_A}\big\downarrow & & \big\downarrow{\scriptstyle \beta_B} \\ A & \xrightarrow{\ \phi\ } & B. \end{array}$$

Definition 4.3. Let (A, β_A) be a graded C*-algebra. The *graded commutator* on A is defined as

$$[a, b] := ab - (-1)^{\deg(a)\deg(b)} ba$$

for all homogeneous elements a, b, and extend by linearality.

Definition 4.4. Let (B, β_B) be a graded C*-algebra. A graded Hilbert B-module is a Hilbert B-module E together with a *linear* map $S_E\colon E \to E$ (called the grading operator) such that $S_E^2 = \mathrm{id}$ and

1. $S_E(eb) = S_E(e)\beta_B(b)$ for all $e \in E$ and $b \in B$.
2. $\langle S_E(e_1), S_E(e_2)\rangle = \beta_B(\langle e_1, e_2\rangle)$ for all $e_1, e_2 \in E$.

Then E decomposes as a direct sum of Banach spaces $E = E_0 \oplus E_1$ where E_i is the $(-1)^i$-eigenspace of S_E, and satisfy

$$E_i B_j \subseteq E_{i+j}, \qquad \langle E_i, E_j\rangle \subseteq B_{i+j}.$$

Example 4.5.

1. The trivial grading on a C*-algebra B is $\beta_B = \mathrm{id}$. Every (ungraded) C*-algebra can be viewed as a trivially graded C*-algebra.
2. Let B be a C*-algebra. The *odd grading* on $B \oplus B$ is given by $\beta_{B\oplus B}(b, b') := (b', b)$.
3. Let (B, β_B) be a graded C*-algebra. It is a graded Hilbert B-module: $S_B = \beta_B$.
4. Let (E, S_E) be a graded Hilbert B-module. The *induced grading* on the C*-algebra $\mathbb{B}_B(E)$ is given by $T \mapsto S_E^{-1} T S_E$.
5. Let (E, S_E) and (F, S_F) be graded Hilbert B-modules. Then $(E \oplus F, S_E \oplus S_F)$ is a graded Hilbert B-module.

Definition 4.6. Let A, B be graded C*-algebras. A *Kasparov* (A, B)*-module* is a triple (E, ϕ, F), where

- E is a graded Hilbert B-module.
- $\phi\colon A \to \mathbb{B}_B(E)$ is a graded *-homomorphism.
- $F \in \mathbb{B}_B(E)$ has degree 1 with respect to the grading induced by E.

satisfying the following "Fredholmness" conditions:

(F1) $[F, \phi(a)] \in \mathbb{K}_B(E)$.

(F2) $(F^2 - 1)\phi(a) \in \mathbb{K}_B(E)$.

(F3) $(F^* - F)\phi(a) \in \mathbb{K}_B(E)$.

for all $a \in A$.

We write $\mathbb{E}(A, B)$ to denote the set of all Kasparov (A, B)-modules.

4.2 Operations on Kasparov modules

4.2.1 Direct sum

Let $\mathcal{E}_i := (E_i, \phi_i, F_i)$ be Kasparov modules, $i = 1, \dots, n$. Then (Example 4.5) $\oplus_i E_i$ is a graded Hilbert C*-module. And $\oplus_i \mathcal{E}_i := (\oplus_i E_i, \oplus_i \phi_i, \oplus_i F_i)$ is a Kasparov module.

4.2.2 Pullback

Let $\mathcal{E} := (E, \phi, F) \in \mathbb{E}(A, B)$ and $f : C \to A$ be a graded *-homomorphism. Then $(E, \phi \circ f, F) \in \mathbb{E}(C, B)$. This is the *pullback* of $\mathcal{E}$ along f, denoted by $f^*\mathcal{E}$.

4.2.3 Interior tensor product

Let $\mathcal{E} := (E, \phi, F) \in \mathbb{E}(A, B)$ and $f : B \to C$ be a graded *-homomorphism. The interior tensor product $\mathcal{E} \otimes_f C \in \mathbb{E}(A, C)$ is defined as $(E \hat{\otimes}_f C, \phi \otimes \mathrm{id}, F \otimes \mathrm{id})$. The grading on $E \hat{\otimes}_f C$ is given by $S_{E\hat{\otimes}_f C}(e \otimes c) := S_E(e) \otimes \beta_C(c)$.

4.2.4 Pushout

Let $\mathcal{E} := (E, \phi, F) \in \mathbb{E}(A, B)$ and $f : B \twoheadrightarrow C$ be a surjective graded *-homomorphism. We can define the pushout $\mathcal{E}_f$ (see Definition 3.26). The pushout $\mathcal{E}_f \in \mathbb{E}(A, C)$ of the Kasparov module $\mathcal{E} \in \mathbb{E}(A, B)$ is defined as (E_f, ϕ_f, F_f) where

- The grading on E_f is defined by $S_{E_f}(q(e)) := q(S_E(e))$.
- $\phi_f(a)q(e) := q(\phi(a)e)$.
- $F_f q(e) := q(Fe)$.

Recall that $q : E \twoheadrightarrow E_f$ is the quotient map.

4.3 Kasparov's KK-group

4.3.1 Homotopies of Kasparov modules

Definition 4.7. Let $\mathcal{E}_i := (E_i, \phi_i, F_i) \in \mathbb{E}(A, B)$, $i = 1, 2$. We say they are isomorphic (write $\mathcal{E}_1 \cong \mathcal{E}_2$), there exists an isomorphism between Hilbert B-modules $\psi : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi(a),\ F} & E_1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ E_2 & \xrightarrow[\phi(a),\ F]{} & E_2 \end{array}$$

Should an isomorphism of Hilbert C*-modules always preserve inner products (hence unitary)?

Example 4.8. Let $\mathcal{E}_i$ be Kasparov modules, $i \in \{1, \dots, n\}$. Let σ be a permutation of $\{1, \dots, n\}$. Then $\oplus_i \mathcal{E}_i \cong \oplus_i \mathcal{E}_{\sigma(i)}$

Let B be a C*-algebra. Let $\mathrm{I}B := \mathrm{C}[0,1] \otimes B \cong \mathrm{C}([0,1], B)$ be the mapping cylinder of B. Let $\mathrm{ev}_t \colon \mathrm{I}B \to B$ be the evaluation map. If B is graded by β_B, then we use $\mathrm{id} \otimes \beta_B$ to grade $\mathrm{I}B$.

Definition 4.9. Let $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A, B)$. A *homotopy* between $\mathcal{E}_0$ and $\mathcal{E}_1$ is $\mathcal{E} \in \mathbb{E}(A, \mathrm{I}B)$ such that $\mathcal{E} \otimes_{\mathrm{ev}_i} B \cong \mathcal{E}_i$ for $i = 0, 1$.

Define the equivalence relation $\sim_{\mathrm{h}}$ on $\mathbb{E}(A, B)$:

$\mathcal{E}_0 \sim_{\mathrm{h}} \mathcal{E}_1$ if they are connected by a finite set of homotopies in $\mathcal{E}(A, \mathrm{I}B)$.

It is not yet clear at the moment whether this is a true equivalence relation: the transitivity holds but we still need to show that $\sim_{\mathrm{h}}$ is reflexive and symmetric. For those we need some lemmas.

Lemma 4.10. *Let $\mathcal{E} \in \mathbb{E}(A, B)$. Let $f\colon B \to C$ and $g\colon C \to D$ be graded *-homomorphisms. Then*

$$(E \otimes_f C) \otimes_g D \cong E \otimes_{g \circ f} D.$$

Proof. Let $\mathcal{E} = (E, \phi, f)$. Define the right D-module map

$$U\colon (E \otimes_f C) \otimes_g D \to E \otimes_{g\circ f} D, \qquad e \otimes_f c \otimes_g d \mapsto e \otimes_{g \circ f} g(c)d.$$

Clearly U is injective. For the surjectivity: take an approximate unit $\{u_i\}$ of B. Then $eu_i \otimes_{g\circ f} d \to e \otimes_{g \circ f} d$, and

$$U(e \otimes_f f(u_i) \otimes_g d) = e \otimes_{g\circ f} g(f(u_i))d = eu_i \otimes_{g\circ f} d \to e \otimes_{g\circ f} d. \qquad \square$$

Lemma 4.11. *Let $\mathcal{E} \in \mathbb{E}(A, B)$ and $f\colon B \twoheadrightarrow C$ be a surjective graded *-homomorphism. Then $\mathcal{E} \otimes_f C \cong \mathcal{E}_f$.*

Remark 4.12. We might have two different notions of compactness while working with homotopies of Kasparov modules. Let E be a Hilbert $\mathrm{I}B$-module and $F \in \mathbb{B}_{\mathrm{I}B}(E)$. Consider the following two statements:

1. $F \in \mathbb{K}_{\mathrm{I}B}(E)$.
2. For each $t \in [0,1]$: $(\mathrm{ev}_t)_* F \in \mathbb{K}_B(E \otimes_{\mathrm{ev}_t} B)$. Here $(\mathrm{ev}_t)_* F$ is the operator $T \otimes_{\mathrm{ev}_t} \mathrm{id}$ acting on $E \otimes_{\mathrm{ev}_t} B$.

In fact: these two conditions are equivalent (Why?). This should be based on the compactness of $[0,1]$, and some property of $\mathbb{K}$ (?). For instance, we have an isomorphism of C*-algebras

$$\mathbb{K}_{\mathrm{I}B}(\mathrm{C}[0,1] \otimes E) \cong \mathrm{C}([0,1], \mathbb{K}_B(E))$$

for a Hilbert B-module E (?) But this does not hold trivially if we replace $\mathbb{K}$ by $\mathbb{B}$. (Why?) I could remember — again read from somewhere — that this holds if we replace the norm topology on $\mathbb{B}$ by the *strict topology*. But I need to find a proof of it. To be completed.

Proposition 4.13. *$\sim_{\mathrm{h}}$ is an equivalence relation.*

Proof.
- By definition, $\sim_{\mathrm{h}}$ is transitive.
- We prove $\sim_{\mathrm{h}}$ is symmetric. If $\mathcal{E} \in \mathbb{E}(A, \mathrm{I}B)$ is a homotopy between $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A, B)$. That is,

$$\mathcal{E} \otimes_{\mathrm{ev}_i} B \cong \mathcal{E}_i, \qquad i = 0, 1.$$

  Define $\psi\colon \mathrm{C}[0,1] \to \mathrm{C}[0,1]$ by $\psi(f)(t) := f(1-t)$. Then $\psi \otimes \mathrm{id}\colon \mathrm{I}B \to \mathrm{I}B$ satisfies $\psi \circ \mathrm{ev}_i = \mathrm{ev}_{1-i}$ for $i = 0, 1$. Therefore

$$\mathcal{E} \otimes_{\psi\otimes\mathrm{id}} \mathrm{I}B \otimes_{\mathrm{ev}_i} B \cong \mathcal{E}_{1-i}, \qquad i = 0, 1.$$

  So $\mathcal{E} \otimes_{\psi\otimes\mathrm{id}} \mathrm{I}B$ is a homotopy between $\mathcal{E}_1$ and $\mathcal{E}_0$.
- Finally, $\sim_{\mathrm{h}}$ is reflexive. Let $\mathcal{E} \in \mathbb{E}(A, B)$. Let $\phi\colon B \to \mathrm{I}B$ be the constant function $\phi(b)(t) := b$. Then $\mathcal{E} \otimes_\phi \mathrm{I}B \in \mathbb{E}(A, \mathrm{I}B)$ is a homotopy betweeen $\mathcal{E}$ and $\mathcal{E}$. $\square$

4.3.2 Operator homotopies of Kasparov modules

Lemma 4.14. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{E}(A,B)$. Let $f\colon B \to C$ be a graded *-homomorphism. Then*

$$(\mathcal{E}_1 \oplus \mathcal{E}_2) \otimes_f C \cong \mathcal{E}_1 \otimes_f C \oplus \mathcal{E}_2 \otimes_f C.$$

Proof. Let $\mathcal{E}_i = (E_i, \phi_i, F_i)$ for $i = 1, 2$. Define

$$U\colon (E_1 \oplus E_2) \otimes_f C \to (E_1 \otimes_f C) \oplus (E_2 \otimes_f C), \qquad (e_1, e_2) \otimes_f c \mapsto (e_1 \otimes_f c, e_2 \otimes_f c).$$

This is an isomorphism of Hilbert C*-modules. It remains to check the Fredholmness conditions (F1), (F2) and (F3) in Kasparov modules, which are easy because the corresponding operators are just direct sums of compact operators, hence compact. □

Corollary 4.15. *Let $\mathcal{E}_0, \mathcal{E}_1, \mathcal{F}_0, \mathcal{F}_1 \in \mathbb{E}(A,B)$ such that $\mathcal{E}_0$ is homotopic to $\mathcal{E}_1$ and $\mathcal{F}_0$ is homotopic to $\mathcal{F}_1$. Then $\mathcal{E}_0 \oplus \mathcal{F}_0$ is homotopic to $\mathcal{E}_1 \oplus \mathcal{F}_1$.*

Proof. Let $\mathcal{E} \in \mathbb{E}(A, IB)$ be a homotopy connecting $\mathcal{E}_0$ and $\mathcal{E}_1$, and $\mathcal{F} \in \mathbb{E}(A, IB)$ be a homotopy connecting $\mathcal{F}_0$ and $\mathcal{F}_1$. Then

$$(\mathcal{E} \oplus \mathcal{F}) \otimes_{\mathrm{ev}_0} B \cong \mathcal{E} \otimes_{\mathrm{ev}_0} B \oplus \mathcal{F} \otimes_{\mathrm{ev}_0} B \cong \mathcal{E}_0 \oplus \mathcal{F}_0,$$

and

$$(\mathcal{E} \oplus \mathcal{F}) \otimes_{\mathrm{ev}_1} B \cong \mathcal{E} \otimes_{\mathrm{ev}_1} B \oplus \mathcal{F} \otimes_{\mathrm{ev}_1} B \cong \mathcal{E}_1 \oplus \mathcal{F}_1.$$

So $\mathcal{E}_0 \oplus \mathcal{F}_0$ is homotopic to $\mathcal{E}_1 \oplus \mathcal{F}_1$ via $\mathcal{E} \oplus \mathcal{F}$. □

Definition 4.16 (Degenerate cycles)**.** Let $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A,B)$. We say $\mathcal{E}$ is *degenerate*, if it satisfies the following “Fredholmness” conditions:

(D1) $[F, \phi(a)] = 0$.

(D2) $(F^2 - 1)\phi(a) = 0$.

(D3) $(F^* - F)\phi(a) = 0$.

for all $a \in A$.

Denote the set of degenerate Kasparov (A,B)-modules by $\mathbb{D}(A,B)$.

Definition 4.17. Let $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A,B)$. We say they are *operator homotopic* if there exists a Hilbert B-module E, $\phi\colon A \to \mathbb{B}_B(E)$ a graded *-homomorphism, and a *norm-continuous* path of operators $F_t \in \mathbb{B}_B(E)$ ($t \in [0,1]$), such that

- $\mathcal{E}_t := (E, \phi, F_t) \in \mathbb{E}(A,B)$ for all $t \in (0,1)$.
- $\mathcal{E}_i \cong (E, \phi, F_i)$ for $i = 0, 1$.

Define the equivalence relation $\sim_{\mathrm{oh}}$ on $\mathbb{E}(A,B)$:

$\mathcal{E}_0 \sim_{\mathrm{oh}} \mathcal{E}_1$ if there exists $\mathcal{F}_0, \mathcal{F}_1 \in \mathbb{D}(A,B)$, such that $\mathcal{E}_0 \oplus \mathcal{F}_0$ is operator homotopic to $\mathcal{E}_1 \oplus \mathcal{F}_1$.

We need to prove that

Proposition 4.18. *$\sim_{\mathrm{oh}}$ is an equivalence relation.*

Proof.
- Reflexivity is obvious: $0 := (0,0,0) \in \mathbb{D}(A,B)$ satisfies $\mathcal{E} \oplus 0 \sim_{\mathrm{oh}} \mathcal{E} \oplus 0$ for any $\mathcal{E} \in \mathbb{E}(A,B)$.
- Symmetry: take the obvious path $\mathcal{E}'_t := \mathcal{E}_{1-t}$.
- Transitivity: let $\mathcal{E}_1 \sim_{\mathrm{oh}} \mathcal{E}_2$ and $\mathcal{E}_2 \sim_{\mathrm{oh}} \mathcal{E}_3$. That is, there exists $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_2, \mathcal{G}_3$ im $\mathbb{D}(A,B)$ such that

  $\mathcal{E}_1 \oplus \mathcal{F}_1$ is operator homotopic to $\mathcal{E}_2 \oplus \mathcal{F}_2$, $\qquad \mathcal{E}_2 \oplus \mathcal{G}_2$ is operator homotopic to $\mathcal{E}_3 \oplus \mathcal{G}_3$.

  Therefore

  $\mathcal{E}_1 \oplus \mathcal{F}_1 \oplus \mathcal{G}_2$ is operator homotopic to $\mathcal{E}_2 \oplus \mathcal{F}_2 \oplus \mathcal{G}_2$,
  is operator homotopic to $\mathcal{E}_3 \oplus \mathcal{G}_3 \oplus \mathcal{F}_2$.

  Notice that the sum of degenerate Kasparov modules is degenerate. So both $\mathcal{F}_1 \oplus \mathcal{G}_2$ and $\mathcal{G}_3 \oplus \mathcal{F}_2$ are degenerate cycles. This finishes the proof. □

4.3.3 Definition of KK-groups

Definition 4.19. According to Proposition 4.13 and Proposition 4.18 we define

$$\mathrm{KK}(A,B) := \mathbb{E}(A,B)/{\sim_{\mathrm{h}}}, \qquad \widehat{\mathrm{KK}}(A,B) := \mathbb{E}(A,B)/{\sim_{\mathrm{oh}}}.$$

Eventually we will show that both of them are abelian groups.

Remark 4.20. We remark on the size issues (arising from the discussion of Marten and Bram). Let B be a C^*-algebra, then the collection of Hilbert B-modules is not a set (for example, consider $B = \mathbb{C}$, then the collection of Hilbert spaces is a class but not a set because for every set we may consider the free vector space generated by this set, equipped with a suitable Hilbert space structure). This happens to Kasparov modules as well.

How to overcome this? In the definition for $\mathbb{E}(A,B)$ we take only *countably-generated* Hilbert B-modules, and the C^*-algebras A and B are also assumed to be separable. Then Kasparov's stablisation theorem claims that every such Hilbert module is a direct summand of $\mathcal{H}_B$. That is, there is an isometry $V\colon E \to \mathcal{H}_B$. We identify E with its image; this identifies E with a submodule of $\mathcal{H}_B$. This identification is allowed because in the definition of KK-groups we will only look at *unitary equivalence classes* of Kasparov modules. The collection of Hilbert B-submodules of $\mathcal{H}_B$ (up to unitary equivalence) forms a set. This allows us to define a group structure at least on the unitary equivalence classes in $\mathbb{E}(A,B)$.

Lemma 4.21. *If $\mathcal{E} \in \mathbb{D}(A,B)$. Then $\mathcal{E} \sim_{\mathrm{h}} 0$.*

Proof. Let $\mathcal{E} = (E,\phi,F)$. As an ideal of the C^*-algebra $\mathrm{C}[0,1]$, $\mathrm{C}_0(0,1]$ is a Hilbert $\mathrm{C}[0,1]$-module, $E \otimes \mathrm{C}_0(0,1]$ denotes the exterior tensor product, which becomes a Hilbert $\mathrm{I}B$-module.

Define

$$\tilde{\mathcal{E}} := (E \hat{\otimes} \mathrm{C}_0(0,1], \phi\otimes \mathrm{id}, F\otimes \mathrm{id}) \in \mathbb{E}(A, \mathrm{I}B).$$

Clearly $\tilde{\mathcal{E}} \otimes_{\mathrm{ev}_0} \mathrm{C}[0,1] = 0$, and notice that there is an isomorphism

$$(E \hat{\otimes} \mathrm{C}_0[0,1)) \otimes_{\mathrm{ev}_1} \mathrm{C}[0,1] \to E, \qquad e \otimes_{\mathrm{ev}_1} f \mapsto f(0)e.$$

Then $\tilde{\mathcal{E}}$ is a homotopy connecting $\mathcal{E}$ and 0. □

Remark 4.22. One should notice that the degeneracy of (E,ϕ,F) is essential: the operator id acting on the Hilbert $\mathrm{C}[0,1]$-module $\mathrm{C}_0(0,1]$ is not compact. So the operators

$$[F,\phi(a)]\otimes \mathrm{id},\ (F^2-1)\phi(a)\otimes \mathrm{id},\ (F^*-F)\phi(a)\otimes \mathrm{id} \in \mathbb{B}_{\mathrm{I}B}(E\otimes \mathrm{C}_0(0,1])$$

are no longer compact unless the Kasparov module (E,ϕ,F) is degenerate.

Lemma 4.23. *Let $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A,B)$. If $\mathcal{E}_0 \sim_{\mathrm{oh}} \mathcal{E}_1$, then $\mathcal{E}_0 \sim_{\mathrm{h}} \mathcal{E}_1$.*

As a corollary, the canonical map $\mu\colon \widehat{\mathrm{KK}}(A,B) \to \mathrm{KK}(A,B)$ is surjective.

Theorem 4.24. *Both $\mathrm{KK}(A,B)$ and $\widehat{\mathrm{KK}}(A,B)$ are abelian groups, and the canonical map $\mu\colon \widehat{\mathrm{KK}}(A,B) \to \mathrm{KK}(A,B)$ is a surjective group homomorphism.*

If A is σ-unital, then μ is an isomorphism.

Proof. We only prove that both $\mathrm{KK}(A,B)$ and $\widehat{\mathrm{KK}}(A,B)$ are abelian groups. For this we need to construct the addition, neutral element and inverse elements. The addition is given by

$$[\mathcal{E}_1] + [\mathcal{E}_2] := [\mathcal{E}_1 \oplus \mathcal{E}_2].$$

Clearly the zero Kasparov module 0 represents the neutral element in both $\mathrm{KK}(A,B)$ and $\widehat{\mathrm{KK}}(A,B)$. Let $\mathcal{E} = (E,\phi,F)$. We construct the inverse of $\mathcal{E}$ as follows.

Let E be graded by S_E and A be graded by β_A. Define

$$E_- := (E, -S_E), \qquad \phi_- := \phi\circ\beta_A, \qquad F_- := -F.$$

and define $-\mathcal{E} := (E_-, \phi_-, F_-)$. We claim that $[-\mathcal{E}]$ is the inverse of $[\mathcal{E}]$: notice that

$$\mathcal{E} \oplus (-\mathcal{E}) = (E \oplus E_-, \phi \oplus \phi \circ \beta_A, F \oplus -F),$$

which is *operator homotopic* to the *degenerate* Kasparov module

$$(E \oplus E_-, \phi \oplus \phi \circ \beta_A, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$$

via the operator homotopy

$$\tilde{F}_t := \begin{pmatrix} \cos \frac{\pi t}{2} F & \sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & -\cos \frac{\pi t}{2} F \end{pmatrix}.$$

So $\mathcal{E} \oplus (-\mathcal{E}) \sim_{\mathrm{oh}} 0$. This also implies $\mathcal{E} \oplus (-\mathcal{E}) \sim_{\mathrm{h}} 0$ by the previous lemma. □

Remark 4.25. Notice that one needs to check that

1. $[\tilde{F}_t, \phi(a) \oplus \phi_-(a)] \in \mathbb{K}_B(E \oplus E_-)$.
2. $(\tilde{F}_t^2 - 1)(\phi(a) \oplus \phi_-(a)) \in \mathbb{K}_B(E \oplus E_-)$.
3. $(\tilde{F}_t - \tilde{F}_t^*)(\phi(a) \oplus \phi_-(a)) \in \mathbb{K}_B(E \oplus E_-)$.

The "most" non-trivial part is the second condition because it is non-linear. This is usually the most difficult condition to check in (the bounded picture of) Kasparov modules. Using unbounded Kasparov modules, however, overcomes this difficulty.

The most prominent construction in KK-theory is the Kasparov product. Its existence and uniqueness was shown by Kasparov [15] in an extremely technical fashion. This was simplified by Connes and Skandalis in [5] by using a "connection" condition, which will be discussed later.

Definition 4.26 (Kasparov product)**.** Let $\mathcal{E}_1 = (E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. A *Kasparov product* of $\mathcal{E}_1$ and $\mathcal{E}_2$ is $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A, C)$ satisfying:

1. $E = E_1 \otimes_{\phi_2} E_2$.
2. $\phi = \phi \otimes_{\phi_2} \mathrm{id}$.
3. F is an F_2-connection. That is,

$$\left[\begin{pmatrix} 0 & T_x^* \\ T_x & 0 \end{pmatrix}, \begin{pmatrix} F_2 & 0 \\ 0 & F \end{pmatrix}\right] \in \mathbb{K}_B(E_2 \oplus E)$$

   for all $x \in E_1$, where $T_x \in \mathbb{B}_B(E_2, E)$ by $T(e_2) = x \otimes_{\phi_2} e_2$.
4. $\phi(a)[F_1 \otimes_{\phi_2} \mathrm{id}, F]\phi(a)^* \geq 0 \mod \mathbb{K}_C(E)$, for all $a \in A$.

Theorem 4.27. *If A is separable. Then the Kasparov product as above exists and is unique up to operator homotopy.*

March 22 and March 29, 2022

KK-theory: Cuntz's picture

Speaker: Yufan Ge (Leiden University)

The standing assumption of this section is that all C*-algebras are *separable* and *σ-unital.*

5.1 Cuntz's $\mathrm{KK_h}$-group

Definition 5.1. Let A and B be C*-algebras. A $\mathrm{KK_h}(A,B)$-cycle is a pair (ϕ_+,ϕ_-), where $\phi_\pm\colon A\to\mathcal{M}(\mathbb{K}\otimes B)$ are *-homomorphisms, such that

$$\phi_+(a)-\phi_-(a)\in\mathbb{K}\otimes B,\qquad \text{for all } a\in A.$$

Denote the set of all $\mathrm{KK_h}(A,B)$-cycles by $\mathbb{F}(A,B)$.

Definition 5.2. Two $\mathrm{KK_h}(A,B)$-cycles (ϕ_+,ϕ_-) and (ψ_+,ψ_-) are *homotopic* if there exists a path of $\mathrm{KK_h}$-cycles $(\lambda_+^t,\lambda_-^t)\in\mathbb{F}(A,B)$, $t\in[0,1]$, continuous in the sense that:

1. $t\mapsto\lambda_\pm^t$ are *strictly continuous.*
2. $t\mapsto\lambda_+^t-\lambda_-^t$ is *norm continuous.*
3. $\lambda_\pm^0=\phi_\pm^0$ and $\lambda_\pm^1=\psi_\pm$.

We write $(\phi_+,\phi_-)\sim_{\mathrm{h}}(\psi_+,\psi_-)$ if they are homotopic.

Remark 5.3. Given a $\mathrm{KK_h}(A,B)$-cycle (ϕ_+,ϕ_-), one can assign to it a Kasparov (A,B)-module

$$\left(\mathcal{H}_B\oplus\mathcal{H}_B,\begin{pmatrix}\phi_+ & \\ & \phi_-\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right).$$

Notice that the Fredholmness condition (F2) implies that

$$\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}\phi_+ & \\ & \phi_-\end{pmatrix}\right]=\begin{pmatrix} & \phi_+-\phi_-\\ \phi_--\phi_+ & \end{pmatrix}$$

is compact. This justifies the condition 2 in the previous definition.

Definition 5.4.

$$\mathrm{KK_h}(A,B):=\mathbb{F}(A,B)/\sim_{\mathrm{h}}.$$

Lemma 5.5. *Let $(\phi_+,\phi_-)\in\mathbb{F}(A,B)$. If $\phi_+=\phi_-$, then $(\phi_+,\phi_-)\sim_{\mathrm{h}}(0,0)$.*

Proof. We need the following

Lemma 5.6. *If B is stable, i.e. $B\cong\mathbb{K}\otimes B$. Then there exists a path $(v_t)_t$ of isometries in $\mathcal{M}(B)$, such that:*

- $t\mapsto v_t$ *is* strictly continuous.
- $v_1=1$.
- $v_tv_t^*\to 0$ *as* $t\to 0$.

Since $\mathbb{K}\otimes B$ is stable, we can choose v_+ and define

$$\lambda_\pm^t:=v_t\phi_\pm v_t^*.$$

Then (ϕ_+^t,ϕ_-^t) is a homotopy connecting (ϕ_+,ϕ_-) and $(0,0)$. ◻

Remark 5.7. We would like to comment that [14, Lemma 1.1.17] is *wrong* unless the *net* $\{m_i\}$ is a *sequence.* This is because a convergence net need not be bounded, so Uniform Boundedness Principle could not be applied to this context.

An example of a convergent net which is not bounded: consider the net $\{x_i\}_{i\in\mathbb{Z}}$ indexed by the direct set $(\mathbb{Z},\geq)$ where $x_i=-i$ if $i<0$ and $x_i=0$ otherwise. The net converges to 0 but it is unbounded since for any $N>0$, $x_{-\lfloor N\rfloor-1}>N$.

Proposition 5.8. *Let B be a stable C^*-algebra. Then there exists an isomorphism*

$$\theta_B\colon \mathbb{M}_n(B) \xrightarrow{\cong} B, \qquad (b_{ij}) \mapsto \sum_{i,j} w_i b_{ij} w_j^*,$$

where $w_i \in \mathcal{M}(B)$ for $i = 1, \ldots, n$ are isometries satisfying

$$w_i^* w_j = \delta_{ij} \qquad \text{and} \qquad \sum_i w_i w_i^* = 1.$$

So such a θ_B is an *inner* isomorphism.

Lemma 5.9. *If B is stable. Then there exists a path v_t of isometries in $\mathcal{M}(B)$, such that*

1. $t \mapsto v_t$ is strictly continuous.
2. $v_1 = 1$.
3. $v_t v_t^* \to 0$ *strictly.*

Corollary 5.10. *If B is stable. Then every isometry in $\mathcal{M}(B)$ is connected to $1 \in \mathcal{M}(B)$ via a strictly continuous path of isometries.*

Proof. Set $w_t = v_t^* w v_t + 1 - v_t v_t^*$. □

Lemma 5.11. *Let B be a stable C^*-algebra. Let $\theta_B\colon \mathbb{M}_2(B) \to B$ as in Proposition 5.8. Let $j\colon B \to \mathbb{M}_2(B)$ be the corner embedding $b \mapsto \left(\begin{smallmatrix} b & 0 \\ 0 & 0\end{smallmatrix}\right)$. Then $\theta_B \circ j$ is homotopic to id_B.*

Proof. By Proposition 5.8, there exists an isometry $w \in \mathcal{M}(B)$ such that $\theta_B \circ j(b) = wbw^*$. By Lemma 5.10, we find a strictly continuous path w_t connecting 1 and w. We conclude that $w_t b w_t^* \to b$ *in norm*: this is because

$$\begin{aligned}\|w_t b w_t^* - w_{t_0} b w_{t_0}^*\| &\leq \|w_t b w_t^* - w_{t_0} b w_t^*\| + \|w_{t_0} b w_t^* - w_{t_0}^* b w_{t_0}^*\| \\ &\leq \|w_t - w_{t_0}\|\|b\| + \|b\|\|w_t^* - w_{t_0}^*\| \to 0\end{aligned}$$

as $t \to t_0$. Notice that we use the fact that w_t are isometries. □

Definition 5.12. Define an addition operation "+" on $\mathrm{KK_h}(A, B)$ as follows:

$$[\phi_+, \phi_-] + [\psi_+, \psi_-] := \left[\theta_B \circ \begin{pmatrix} \phi_+ & \\ & \psi_+ \end{pmatrix}, \theta_B \circ \begin{pmatrix} \phi_- & \\ & \psi_- \end{pmatrix}\right].$$

Proposition 5.13. *The addition is well-defined and turns $\mathrm{KK_h}(A, B)$ into an abelian group with neutral element given by $[0, 0]$.*

Proof.
- Well-definedness: this is done by checking a homotopy.
- Associativity. We have

$$\begin{aligned}\theta_B \begin{pmatrix} \theta_B \begin{pmatrix} \phi_+ & \\ & \psi_+ \end{pmatrix} & \\ & \lambda_+ \end{pmatrix} &\sim_{\mathrm{h}} \theta_B \begin{pmatrix} \theta_B \begin{pmatrix} \phi_+ & \\ & \psi_+ \end{pmatrix} & \\ & \theta_B \begin{pmatrix} \lambda_+ & \\ & 0 \end{pmatrix} \end{pmatrix} \\ &= \theta_B(\mathrm{id} \otimes \theta_B) \begin{pmatrix} \phi_+ & & & \\ & \psi_+ & & \\ & & \lambda_+ & \\ & & & 0 \end{pmatrix} \\ &\sim_{\mathrm{h}} \theta_B(\mathrm{id} \otimes \theta_B) \begin{pmatrix} \phi_+ & & & \\ & 0 & & \\ & & \psi_+ & \\ & & & \lambda_+ \end{pmatrix} \\ &\sim_{\mathrm{h}} \theta_B \begin{pmatrix} \phi_+ & \\ & \theta_B \begin{pmatrix} \psi_+ & \\ & \lambda_+ \end{pmatrix} \end{pmatrix}\end{aligned}$$

So
$$([\phi_+, \phi_-] + [\psi_+, \psi_-]) + [\lambda_+, \lambda_-] = [\phi_+, \phi_-] + ([\psi_+, \psi_-] + [\lambda_+, \lambda_-]).$$

- Neutral element and inverse. Notice that
$$\begin{aligned}
[\phi_+, \phi_-] + [\phi_-, \phi_+] &= \left[\theta_B \circ \begin{pmatrix} \theta_+ & \\ & \theta_- \end{pmatrix}, \theta_B \circ \begin{pmatrix} \theta_- & \\ & \theta_+ \end{pmatrix}\right] \\
&\sim_{\mathrm{h}} \left[\theta_B \circ \begin{pmatrix} \theta_+ & \\ & \theta_- \end{pmatrix}, \theta_B \circ \begin{pmatrix} \theta_+ & \\ & \theta_- \end{pmatrix}\right] \\
&\sim_{\mathrm{h}} [0, 0].
\end{aligned}$$
□

5.2 From quasihomomorphisms to Kasparov modules

Let $\Psi \colon \mathcal{M}(\mathbb{K} \otimes B) \xrightarrow{\cong} \mathbb{B}_B(\mathcal{H}_B)$ be the isomorphism that extends the isomorphism
$$\mathbb{K} \otimes B \xrightarrow{\cong} \mathbb{K}_B(\mathcal{H}_B), \qquad e_{ij} \otimes ab^* \mapsto \Theta_{a_i, b_j}. \tag{2}$$

For $(\phi_+, \phi_-) \in \mathbb{F}(A, B)$: let $\hat{\mathcal{H}}_B$ be the evenly graded Hilbert B-module $\mathcal{H}_B \oplus \mathcal{H}_B$. Define the associated Kasparov module
$$\mathcal{E}(\phi_+, \phi_-) := \left(\hat{\mathcal{H}}_B, \begin{pmatrix} \Psi \circ \phi_+ & \\ & \Psi \circ \phi_- \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \in \mathbb{E}(A, B).$$

Lemma 5.14. *The map*
$$\mathrm{KK}_{\mathrm{h}}(A, B) \to \mathrm{KK}(A, B), \qquad [\phi_+, \phi_-] \mapsto [\mathcal{E}(\phi_+, \phi_-)]$$
is a group homomorphism.

Proof.
- We first prove it is well-defined. Let $(\lambda_+^t, \lambda_-^t)$ be a homotopy connecting (ϕ_+, ϕ_-) and (ψ_+, ψ_-). Define
$$\lambda_\pm \colon A \to \mathcal{M}(\mathrm{I}(\mathbb{K} \otimes B)), \qquad (\lambda_\pm(a) f)(t) := \lambda_\pm^t(a)(f(t))$$
for $f \in \mathrm{I}(\mathbb{K} \otimes B) \cong \mathbb{K} \otimes \mathrm{I}B$. So $(\lambda_+, \lambda_-) \in \mathbb{F}(A, \mathrm{I}B)$.
We claim that $(\mathrm{ev}_t)_* \mathcal{E}(\lambda_+, \lambda_-) \cong \mathcal{E}(\lambda_+^t, \lambda_-^t)$ for $t \in [0, 1]$. We have an isomorphism of graded Hilbert B-modules
$$\chi \colon (\mathrm{ev}_t)_* \hat{\mathcal{H}}_{\mathrm{I}B} \xrightarrow{\cong} \hat{\mathcal{H}}_B, \qquad (x, y) \otimes_{\mathrm{ev}_t} B \mapsto ((\mathrm{ev}_t(x_i)), (\mathrm{ev}_t(y_i))).$$
To prove that two Kasparov modules above are homotopic it suffices to check the following diagram commutes:
$$\begin{array}{ccc}
(\mathrm{ev}_t)_* \hat{\mathcal{H}}_{\mathrm{I}B} & \xrightarrow{(\mathrm{ev}_t)_* \begin{pmatrix} \Psi \circ \lambda_+(a) & \\ & \Psi \circ \lambda_-(a) \end{pmatrix}} & (\mathrm{ev}_t)_* \hat{\mathcal{H}}_{\mathrm{I}B} \\
\Big\downarrow \chi & & \Big\downarrow \chi \\
\hat{\mathcal{H}}_B & \xrightarrow{\begin{pmatrix} \Psi \circ \lambda_+^t(a) & \\ & \Psi \circ \lambda_-^t(a) \end{pmatrix}} & \hat{\mathcal{H}}_B.
\end{array}$$
And this is done by checking on the elements in $\mathbb{M}_2(\mathbb{K} \otimes \mathrm{I}B)$, which form a stricly dense subset of $\mathbb{M}_2(\mathcal{M}(\mathbb{K} \otimes \mathrm{I}B))$.
- Next we show that μ is a group homomorphism. For this we need to construct an isomorphism
$$[\mathcal{E}((\phi_+, \phi_-) + (\psi_+, \psi_-))] \xrightarrow{\cong} [\mathcal{E}(\phi_+, \phi_-)] + [\mathcal{E}(\psi_+, \psi_-)],$$
as follows. Let $v_1, v_2 \in \mathcal{M}(\mathbb{K} \otimes B)$ be isometries. Write $w_i = \Psi(v_i)$ and define the isomorphism
$$\hat{\mathcal{H}}_B \oplus \hat{\mathcal{H}}_B \xrightarrow{\cong} \hat{\mathcal{H}}_B, \qquad (x_1, x_2) \oplus (y_1, y_2) \mapsto (w_1 x_1 + w_2 y_1, w_1 x_2 + w_2 y_2).$$

This isomorphism gives the following homotopies:

$$
\begin{aligned}
\mathcal{E}(\phi_+,\phi_-)+\mathcal{E}(\psi_+,\psi_-) &= \left(\hat{\mathcal{H}}_B \oplus \hat{\mathcal{H}}_B, \begin{pmatrix} \begin{pmatrix} \Psi\circ\phi_+ & \\ & \Psi\circ\phi_- \end{pmatrix} & \\ & \begin{pmatrix} \Psi\circ\psi_+ & \\ & \Psi\circ\psi_- \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & \\ & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{pmatrix} \right) \\
&\sim_{\mathrm{h}} \left(\hat{\mathcal{H}}_B, \begin{pmatrix} \Psi\circ\theta_B \begin{pmatrix} \phi_+ & \\ & \psi_+ \end{pmatrix} & \\ & \Psi\circ\theta_B \begin{pmatrix} \phi_- & \\ & \psi_- \end{pmatrix} \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right) \\
&= \mathcal{E}\left(\theta_B \circ \begin{pmatrix} \phi_+ & \\ & \psi_+ \end{pmatrix}, \theta_B \circ \begin{pmatrix} \phi_- & \\ & \psi_- \end{pmatrix}\right) \\
&= \mathcal{E}((\phi_+,\phi_-)+(\psi_+,\psi_-)).
\end{aligned}
$$

□

Remark 5.15. The map $\mathrm{KK_h}(A,B) \to \mathrm{KK}(A,B)$, $[\phi_+,\phi_-] \mapsto [\mathcal{E}(\phi_+,\phi_-)]$ is in fact an isomorphism.

5.3 Functoriality

5.3.1 Pullback

The pullback is easily formed.

Definition 5.16. Let $f\colon D \to A$ be a *-homomorphism. The pullback f^* is the induced group homomorphism

$$f^*\colon\ \mathrm{KK_h}(A,B) \to \mathrm{KK_h}(D,B), \qquad f^*([\phi_+,\phi_-]) := [\phi_+\circ f, \phi_-\circ f].$$

5.3.2 Pushout

The pushout needs more work.

Definition 5.17. A *-homomorphism $\phi\colon A \to B$ is called *quasi-unital* if there exists $p \in \mathcal{M}(B)$ such that $\overline{\phi(A)B} = pB$.

This is equivalent to: if the hereditory subalgebra of B generated by $\phi(A)$ is pBp for some $p \in \mathcal{M}(B)$.

Let $\mathrm{Hom_{qu}}(A,B)$ denote the set of quasi-unital *-homomorphisms $A \to B$.

Lemma 5.18 (Higson). *A *-homomorphism $\phi\colon A \to B$ if quasi-unital iff there exists a *-homomorphism $\phi\colon \mathcal{M}(A) \to \mathcal{M}(B)$ extending ϕ.*

Corollary 5.19. *The composition of two quasi-unital *-homomorphisms is quasi-unital.*

We introduce a stronger homotopy relation $\sim_{\mathrm{qu}}$ on $\mathrm{Hom_{qu}}(A,B)$.

Definition 5.20. We call $\phi_0,\phi_1 \in \mathrm{Hom_{qu}}(A,B)$ strongly homotopic (denoted by $\phi_0 \sim_{\mathrm{qu}} \phi_1$), if there exists $\phi \in \mathrm{Hom_{qu}}(A, IB)$ such that $\mathrm{ev}_i \circ\phi = \phi_i$, $i = 0,1$.

Recall that $\phi_0 \sim \phi_1$ denotes ϕ_0 and ϕ_1 are homotopic. Define

$$[A,B] := \mathrm{Hom}(A,B)/{\sim}, \qquad [A,B]_{\mathrm{qu}} := \mathrm{Hom_{qu}}(A,B)/{\sim_{\mathrm{qu}}}\ .$$

Theorem 5.21. *If A and B are σ-unital C^*-algebras and B is stable. Then there is a bijection*

$$[A,B]_{\mathrm{qu}} \xrightarrow{\cong} [A,B]$$

(sending the strongly homotopy class of a quasi-unital *-homomorphism in $[A,B]_{\mathrm{qu}}$ to its homotopy class in $[A,B]$).

If $f\colon \mathbb{K}\otimes B \to \mathbb{K}\otimes C$ be a quasi-unital *-homomorphism. By Lemma 5.18, f extends to a *-homomorphism $\mathcal{M}(\mathbb{K}\otimes B) \to \mathcal{M}(\mathbb{K}\otimes C)$.

Definition 5.22. The pushout f_* is the induced group homomorphism

$$\mathrm{KK_h}(A, B) \longrightarrow \mathrm{KK_h}(A, C), \qquad [\phi_+, \phi_-] \mapsto [f \circ \phi_+, f \circ \phi_-].$$

Lemma 5.23. *Let $(\phi_+, \phi_-), (\psi_+, \psi_-) \in \mathbb{F}(A, B)$ and assume $\phi_+(a)\psi_+(a) = 0 = \phi_-(a)\psi_-(a)$. Then $[\phi_+, \phi_-] + [\psi_+, \psi_-] = [\phi_+ + \psi_+, \phi_- + \psi_-]$.*

Proof. Define

$$\lambda_\pm^t := \theta_B \begin{pmatrix} \phi_\pm & 0 \\ 0 & 0 \end{pmatrix} + \theta \circ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \psi_\pm \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

This is a homotopy connecting $\left(\theta_B \begin{pmatrix} \phi_+ & \\ & \psi_+ \end{pmatrix}, \theta_B \begin{pmatrix} \phi_- & \\ & \psi_- \end{pmatrix}\right)$ and $(\phi_+ + \psi_+, \phi_- + \psi_-)$. □

Lemma 5.24. *Let $(\phi_+, \phi_-) \in \mathbb{F}(A, B)$. Let $w \in \mathcal{M}(\mathbb{K} \otimes B)$ be partial isometries such that $w^*w \geq \phi_\pm(1)$. Then $[w\phi_+w^*, w\phi_-w^*] = [\phi_+, \phi_-]$ in $\mathrm{KK_h}(A, B)$.*

Proof. Set

$$S_t := \begin{pmatrix} \cos \frac{\pi}{2}t & -w^* \sin \frac{\pi}{2}t \\ w \sin \frac{\pi}{2}t & ww^* \cos \frac{\pi}{2}t \end{pmatrix}.$$

Then the map

$$S_t^* \begin{pmatrix} \phi_+(\cdot) & \\ & w\phi_-(\cdot)w^* \end{pmatrix} S_t$$

is a *-homomorphism. The homotopy between the two $\mathrm{KK_h}$-cycles is therefore given by $(\lambda_+^t, \lambda_-^t)$ where

$$\lambda_+ := \theta_B \begin{pmatrix} \phi_- & \\ & w\phi_+w^* \end{pmatrix}, \qquad \lambda_- := \theta_B \left(S_t^* \begin{pmatrix} \theta_+ & \\ & w\phi_-w^* \end{pmatrix} S_t \right).$$ □

Proposition 5.25. *The pushout $f_* \colon \mathrm{KK_h}(A, B) \longrightarrow \mathrm{KK_h}(A, C)$ is a homomorphism.*

Proof. Let (ϕ_+, ϕ_-) and $(\psi_+, \psi_-) \in \mathbb{F}(A, B)$. Let $v_1, v_2 \in \mathcal{M}(\mathbb{K} \otimes B)$ be isometries given in Proposition 5.8. Then

$$\begin{aligned} f_*([\phi_+, \phi_-] + [\psi_+, \psi_-]) &= f_*([v_1\phi_+v_1^* + v_2\phi_+v_2^*, v_1\phi_-v_1^* + v_2\phi_-v_2^*]) \\ &= [\mathrm{Ad}_{f(v_1)} f(\phi_+) + \mathrm{Ad}_{f(v_2)} f(\psi_+), \dots] \end{aligned}$$

Use $v_1^*v_2 = 0$ (Proposition 5.8) and Lemma 5.23:

$$= [\mathrm{Ad}_{f(v_1)} f(\phi_+), \mathrm{Ad}_{f(v_2)} f(\phi_-)] + [\dots]$$

Use Lemma 5.24:

$$= f_*[\phi_+, \phi_-] + f_*[\psi_+, \psi_-].$$ □

Corollary 5.26.

- Functoriality. *$(f \circ g)_* = f_* \circ g_*$ for composable quasi-unital *-homomorphisms f and g.*
- *If $f_0 \sim_{\mathrm{qu}} f_1$. Then $f_{0*} = f_{1*}$.*

Theorem 5.27. $\mathrm{KK_h}(A, \cdot) \colon \mathsf{C^*Alg} \longrightarrow \mathsf{Ab}$ *is a stable functor. That is, the corner embedding $B \hookrightarrow B \otimes \mathbb{K}$ induces an isomorphism*

$$\mathrm{KK_h}(A, B) \cong \mathrm{KK_h}(A, B \otimes \mathbb{K}).$$

April 5, 2022

Properties and examples of KK-theory

Speaker: Yuezhao Li (Leiden University)

6.1 What is KK-theory?

We have seen the definitions of KK-groups formulated in Kasparov's picture and Cuntz's picture. It might be meaningful to ask what is KK-theory and what it is supposed to do. Depending on the viewpoint there are several answers:

- A theory of generalised pseudodifferential operators (Kasparov).
- Generalised homomorphisms between C^*-algebras (Cuntz).
- A bivariant theory/bifunctor of (a suitable full subcategory of) C^*-algebras which generalise both K-theory and K-homology (Atiyah, Kasparov). That is, for a C^*-algebra A:

$$\mathrm{KK}(\mathbb{C}, A) \cong \mathrm{K}_0(A), \qquad \mathrm{KK}(A, \mathbb{C}) \cong \mathrm{K}^0(A),$$

  where $\mathrm{K}^0(A)$ is the K-*homology* of A.
- A category of (a suitable class of) C^*-algebras which has very nice structure (e.g. a triangulated structure) (Higson, Meyer–Nest, Mesland).

These different viewpoints are, of course, related to each other. Each of them can be more powerful and useful in a certain context. I will explain some of the relations between them by examining some examples; in a future talk of mine I will explain some categorical aspect of KK-theory.

6.2 Examples of Kasparov modules

6.2.1 Kasparov modules from *-homomorphisms

As indicated by Cuntz's picture, KK-theory can be viewed as generalised homomorphisms between C^*-algebras. The usual *-homomorphisms should also represent elements in KK-theory. This is true.

Example 6.1. Let $f\colon A \to B$ be a (graded) *-homomorphism. Then $\mathbb{K}_B(B) \cong B$. So f yields a Kasparov module $[f] := (B, f, 0) \in \mathrm{KK}(A, B)$.

In particular: $\mathrm{id}_A\colon A \to A$ defines a class $[\mathrm{id}_A]$ in $\mathrm{KK}(A, A)$. This is a special element: $\mathrm{KK}(A, A)$ equipped with the Kasparov product becomes a ring, and $[\mathrm{id}_A]$ is the unit element.

6.2.2 K-theory

Theorem 6.2. *Let B be a C^*-algebra, viewed as a trivially graded C^*-algebra. Then* $\mathrm{KK}(\mathbb{C}, B) \cong \mathrm{K}_0(B)$.

Proof. Assume B is unital. Let $p \in \mathbb{M}_n(B)$ be a projection. Then pB^n is a finitely-generated projective B-module. The converse is also true: any finitely-generated projective B-module gives a projection in $\mathbb{M}_n(B)$ by projecting B^n onto this projective module. We need a graded Hilbert B-module in the Kasparov module. So we define

$$E_p := pB^n \oplus pB^n$$

equipped with the even grading. Let $\pi\colon \mathbb{C} \to \mathbb{B}_B(E_B)$ be the unital inclusion. Then $(E_p, \pi, 0) \in \mathbb{E}(\mathbb{C}, B)$ and hence defines a class in $\mathrm{KK}(\mathbb{C}, B)$. This correspondence yields a map

$$\mathrm{K}_0(B) \to \mathrm{KK}(\mathbb{C}, B), \qquad [p] \mapsto [E_p, \pi, 0],$$

which is well-defined.

For the converse direction, starting with $(E, \phi, F) \in \mathbb{E}(\mathbb{C}, B)$ we want to associate a K-theory class to it. This K-theory class should give a Kasparov module of the form $(E_p, \pi, 0)$ where E_p is finitely-generated, projective B-module and π is the unital inclusion. We use several steps to build a homotopy between (E, ϕ, F) and such a Kasparov module.

- Without changing the homotopy class of (E, ϕ, F) we may assume that ϕ is unital. This is done as follows: let $q := \phi(1)$. Then q is a projection in $\mathbb{B}_B(E)$ and ϕ maps to its range becauuse $\phi(a) = \phi(1)\phi(a)\phi(1)$. Then we may consider (qE, ϕ, qFq), which belongs to the same homotopy class of (E, ϕ, F): notice that

$$\begin{aligned}(E, \phi, F) &\sim_{\mathrm{oh}} (qE \oplus (1-q)E, \phi \oplus 0, qFq \oplus (1-q)F(1-q)) \\ &= (qE, \phi, qFq) + ((1-q)E, 0, (1-q)F(1-q)).\end{aligned}$$

  But $((1-q)E, 0, (1-q)F(1-q))$ is degenerate.
- Then we may assume $\phi \colon \mathbb{C} \to \mathbb{B}_B(E)$ is unital, so it is the unital inclusion $\pi \colon \mathbb{C} \hookrightarrow \mathbb{B}_B(E)$. To build a Kasparov module of the form $(E', \pi, 0)$, we need to let F have closed image. This does not always happen even though F is a "generalised Fredholm operator" because operators on Hilbert C^*-modules are usually quite weird. But we may find a compact perturbation G of F satisfying this property. That is, consider the image $\overline{F}$ in the Calkin algebra $\mathbb{B}_B(E)/\mathbb{K}_B(E)$. The conditions (F2) and (F3) for a Kasparov module imply that

$$\overline{F}^2 = 1, \qquad \overline{F} = \overline{F}^*,$$

  that is, $\overline{F} \in \mathbb{B}_B(E)/\mathbb{K}_B(E)$ is a self-adjoint unitary. It lifts to a self-adjoint partial isometry $G \in \mathbb{B}_B(E)$ (see [24, Lemma 17.1.2]). Then G has closed image and $G - F \in \mathbb{K}_B(E)$. So (E, π, G) lies in the same homotopy class with (E, π, F).
- We obtain the Kasparov module $(\ker G, \pi, 0)$. We claim that $\ker G$ is finitely-generated and projective:
  - The image of G in $\mathbb{B}/\mathbb{K}$ is $\overline{F}$, hence G is invertible modulo compact. Let its parametrix be G', then $G'G - \mathrm{id} \in \mathbb{K}$. Restricting to $\ker G$ we have $\mathrm{id}_{\ker G} \in \mathbb{K}$, so $\ker G$ is finitely-generated.
  - We have $E \cong \ker G \oplus \operatorname{im} G$. Since $\operatorname{im} G$ is closed, $\ker G$ is a direct summand in E and hence projective.

  Finally, the Kasparov module $(\ker G, \pi, 0)$ is homotopic to (E, π, G) via $(\tilde{E}, \tilde{\pi}, \tilde{F})$, where

$$\tilde{E} := \{f \in \mathrm{I}E \mid f(1) \in \ker F\}, \quad \tilde{\pi} := \pi \otimes \mathrm{id}, \quad \tilde{F} := F \otimes \mathrm{id}. \qquad \square$$

6.2.3 K-homology

The idea of K-homology originates from Atiyah. Topological K-theory, established by Atiyah and Hirzebruch, is a generalised cohomology theory of spaces. Imposing some duality isomorphisms, there is a dual theory (in a suitable sense) of K-theory, called K-*homology*. Let X be a locally compact topological space, we write $\mathrm{K}_0(X)$ for its K-homology

Atiyah observed that this dual theory can be described by elliptic operators: elements of K-homology groups can be represented by the so-called generalised elliptic operators, called K-cycles. There is a "index pairing" between generalised elliptic operators and vector bundles, mapping to integers. All these were made clear by Kasparov: a K-cycle for $\mathrm{K}_0(X)$ is a Kasparov $(\mathrm{C}_0(X), \mathbb{C})$-module, and $\mathrm{K}_0(X) \cong \mathrm{KK}(\mathrm{C}_0(X), \mathbb{C})$.

Example 6.3. Let X be a smooth closed manifold. Let E and E' be vector bundles over X. Choosing a partition of unity, we may define a Riemannian (or Hermitian) metric on M, and $L^2(X, E)$ and $L^2(X, E')$, the space of L^2-sections of these two vector bundles. Then $\mathrm{C}(X)$ acts on $L^2(X, E) \oplus L^2(X, E')$ by multiplication. Denote this multiplication action by π. Let $P \colon \mathrm{C}^\infty(E) \to \mathrm{C}^\infty(E')$ be elliptic. Then it has a parametrix $Q \colon \mathrm{C}^\infty(E') \to \mathrm{C}^\infty(E)$. Both P and Q extends to (essentially unitary, Fredholm) operators on the L^2-spaces of sections, which are bounded because they are of order 0. Then

$$\left(L^2(X, E) \oplus L^2(X, E'), \pi, \begin{pmatrix} & Q \\ P & \end{pmatrix}\right)$$

defines an element in $\mathbb{E}(\mathrm{C}(X), \mathbb{C})$.

6.3 Properties of KK-theory

6.3.1 Functoriality

We restrict to the full subcategory of separable C^*-algebras $\mathrm{C}^*\mathrm{Sep}$.

Proposition 6.4. KK *is a bifunctor* $\mathrm{C}^*\mathrm{Sep}^{\mathrm{op}} \times \mathrm{C}^*\mathrm{Sep} \to \mathrm{Ab}$.

Proving the proposition is easy: we just need to show that the functoriality of Kasparov modules (Section 4.2) descend to KK-theory. Then it suffices to check that the functorial operations preserve homotopy relations and direct sums.

Let $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A, B)$.

Pullback Let $f: A' \to A$ be a *-homomorphism. Recall that the pullback of $\mathcal{E}$ along f is

$$f^*\mathcal{E} := (E, \phi \circ f, F) \in \mathbb{E}(A', B).$$

The direct sums are preserved: $f^*(\mathcal{E}+\mathcal{E}') = f^*\mathcal{E}+f^*\mathcal{E}'$. If $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A, B)$ are homotopic through $\mathcal{E} \in \mathbb{E}(A, \mathrm{I}B)$. Then $f^*\mathcal{E} \in \mathbb{E}(A', \mathrm{I}B)$ is a homotopy connecting $f^*\mathcal{E}_0$ and $f^*\mathcal{E}_1$. Therefore, f defines a map $\mathrm{KK}(A, B) \to \mathrm{KK}(A', B)$.

Pushout Let $g: B \to B'$ be a *-homomorphism. Recall that the pullback of $\mathcal{E}$ along g is

$$g_*\mathcal{E} := (E \otimes_g B', \phi \otimes_g \mathrm{id}, F \otimes_g \mathrm{id}) \in \mathbb{E}(A, B').$$

The direct sums are preserved: $g_*(\mathcal{E}+\mathcal{E}') = g_*\mathcal{E}+g_*\mathcal{E}'$. If $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A, B)$ are homotopic through $\mathcal{E} \in \mathbb{E}(A, \mathrm{I}B)$. Then there is a homotopy connecting $g_*\mathcal{E}_0$ and $g_*\mathcal{E}_1$ given by $(\mathrm{I}g)_*\mathcal{E} \in \mathbb{E}(A, \mathrm{I}B')$. Here $\mathrm{I}g := g \otimes \mathrm{id}: \mathrm{I}B \to \mathrm{I}B'$.

We introduce another operation on KK-theory:

Suspension Let D be a C^*-algebra. Define the Kasparov module

$$\tau_D(\mathcal{E}) := (E \otimes D, \phi \otimes \mathrm{id}, F \otimes \mathrm{id}) \in \mathbb{E}(A \otimes D, B \otimes D).$$

Here $E \otimes D$ is the *exterior* tensor product of E and D (viewed as a Hilbert D-module).

"Slogan" I would like to remark the following slogan:

> "All constructions are Kasparov products."

Proposition 6.5. *Let* $\mathcal{E} \in \mathbb{E}(A, B)$.

- *Let* $f: A' \to A$ *be a* **-homomorphism. Then* $[f^*\mathcal{E}] = [f] \otimes_A [\mathcal{E}]$.
- *Let* $g: B \to B'$ *be a* **-homomorphism. Then* $[g_*\mathcal{E}] = [\mathcal{E}] \otimes_B [g]$.

The proposition can be checked by using the Connes–Skandalis conditions for connections (Definition 4.26).

6.3.2 Homotopy invariance

Theorem 6.6.
- *If* $f_0, f_1: A' \rightrightarrows A$ *are homotopic* **-homomorphisms. Then* $f_0^* = f_1^*: \mathrm{KK}(A, B) \to \mathrm{KK}(A', B)$ *for any* C^**-algebra* B.
- *If* $g_0, g_1: B \rightrightarrows B'$ *are homotopic* **-homomorphisms. Then* $g_{0*} = g_{1*}: \mathrm{KK}(A, B) \to \mathrm{KK}(A, B')$ *for any* C^**-algebra* A.

Proof.

- If $f_0, f_1 \colon A' \rightrightarrows A$ are homotopic *-homomorphisms. This means there is a *-homomorphism
$$f \colon A' \to \mathrm{I}A$$
such that $\mathrm{ev}_i \circ f = f_i$ for $i = 0, 1$. Given any class in $\mathrm{KK}(A, B)$, choose a representative $\mathcal{E} \in \mathbb{E}(A, B)$. The image of $\mathcal{E}$ under the map
$$\mathbb{E}(A, B) \xrightarrow{\tau_{\mathrm{C}[0,1]}} \mathbb{E}(\mathrm{I}A, \mathrm{I}B) \xrightarrow{f^*} \mathbb{E}(A', \mathrm{I}B)$$
defines a homotopy between $f_0^*\mathcal{E}$ and $f_1^*\mathcal{E}$.
- If $g_0, g_1 \colon B \rightrightarrows B'$ are homotopic *-homomorphisms. This means there is a *-homomorphism $g \colon B \to \mathrm{I}B'$ such that $\mathrm{ev}_i \circ g = g_i$ for $i = 0, 1$. Given any class in $\mathrm{KK}(A, B)$, choose a representative $\mathcal{E} \in \mathbb{E}(A, B)$. Then $g_*\mathcal{E} \in \mathbb{E}(A, \mathrm{I}B')$ defines a homotopy between $g_{0*}\mathcal{E}$ and $g_{1*}\mathcal{E}$. □

6.3.3 Stability

Theorem 6.7. *The map* $\tau_{\mathbb{K}} \colon \mathrm{KK}(A, B) \to \mathrm{KK}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$ *is an isomorphism.*

Proof. We construct an inverse of $\tau_{\mathbb{K}}$. Let $\mathcal{E} \in \mathbb{E}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$. Then

$$(\mathcal{E} \otimes_\Psi \mathcal{H}_B, (\phi \circ e) \otimes_\Psi \mathrm{id}, F \otimes_\Psi \mathrm{id}) \in \mathbb{E}(A, B),$$

where $\Psi \colon B \otimes \mathbb{K} \to \mathbb{K}_B(\mathcal{H}_B)$ is the isomorphism in (2) and $e \colon A \hookrightarrow A \otimes \mathbb{K}$ is a corner embedding. This defines an inverse map for $\tau_{\mathbb{K}}$. □

Corollary 6.8. KK *is* $\mathbb{K}$*-stable in both variables, i.e. the corner embeddings induce isomorphisms*

$$\mathrm{KK}(A, B) \cong \mathrm{KK}(A \otimes \mathbb{K}, B) \cong \mathrm{KK}(A, B \otimes \mathbb{K}).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{KK}(A, B) & \longrightarrow & \mathrm{KK}(A \otimes \mathbb{K}, B) \\ \downarrow & \searrow^{\tau_{\mathbb{K}}} & \downarrow{\scriptstyle \tau_{\mathbb{K}}} \\ \mathrm{KK}(A, B \otimes \mathbb{K}) & \xrightarrow[\tau_{\mathbb{K}}]{} & \mathrm{KK}(A \otimes \mathbb{K}, B \otimes \mathbb{K}). \end{array}$$

Here we use $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$. Notice that the $\tau_{\mathbb{K}}$'s in the diagram are all isomorphisms. Hence all arrows in the diagram are isomorphisms. □

6.3.4 Bott periodicity

Definition 6.9. An element $x \in \mathrm{KK}(A, B)$ is called a KK-*equivalence*, if there exists $y \in \mathrm{KK}(B, A)$ such that

$$x \otimes_B y = [\mathrm{id}_A] \in \mathrm{KK}(A, A), \qquad y \otimes_A x = [\mathrm{id}_B] \in \mathrm{KK}(B, B).$$

Two C*-algebras A and B are KK-*equivalent* (denoted by $A \sim_{\mathrm{KK}} B$), if there exists a KK-equivalence $x \in \mathrm{KK}(A, B)$.

Remark 6.10. Note that the definition above is very similar to the definition of an isomorphism. This is true: two C*-algebras are KK-equivalent if they are isomorphic in the Kasparov category KK.

The proof of the following theorem is left to future talks:

Theorem 6.11. *For any* C*-*algebra* A, $A \sim_{\mathrm{KK}} \mathrm{S}^2 A$. *In particular:* $\mathbb{C} \sim_{\mathrm{KK}} \mathrm{C}_0(\mathbb{R}^2)$.

Corollary 6.12. *For any* C*-*algebra* D *and* A, $\mathrm{KK}(D, A) \cong \mathrm{KK}(D, \mathrm{S}^2 A)$ *and* $\mathrm{KK}(A, D) \cong \mathrm{KK}(\mathrm{S}^2 A, D)$.

Proof. Consider

$$\mathrm{KK}(D,A) \underset{\cdot\otimes y}{\overset{\cdot\otimes x}{\rightleftarrows}} \mathrm{KK}(D,\mathrm{S}^2A)$$

where $x \in \mathrm{KK}(A,\mathrm{S}^2A)$ and $y \in \mathrm{KK}(\mathrm{S}^2A,A)$ are KK-equivalences such that $x\otimes y = \mathrm{id}_A$ and $y\otimes x = \mathrm{id}_{\mathrm{S}^2A}$. Let $z \in \mathrm{KK}(D,A)$. Notice that under the map

$$\mathrm{KK}(D,A) \xrightarrow{\cdot\otimes x} \mathrm{KK}(D,\mathrm{S}^2A) \xrightarrow{\cdot\otimes y} \mathrm{KK}(D,A),$$

The element z is sent to $z\otimes x\otimes y = z\otimes(x\otimes y) = z\otimes[\mathrm{id}_A] = (\mathrm{id}_A)_*z = z$ (Proposition 6.5). A similar result holds for the composition on the inverse direction. Therefore $\cdot\otimes x$ and $\cdot\otimes y$ are inverses to each other, hence both group isomorphisms. □

6.3.5 Long exact sequence

Given an extension of C*-algebra, one might hope to have a long exact sequence in KK-theory. Unfortunately, this is not the case in general. But for semi-split extensions, we do have induced long exact sequences.

Recall that a linear map $f\colon A\to B$ between C*-algebras is

- *completely positive* if $f\otimes\mathrm{id}\colon A\otimes\mathbb{K}\to B\otimes\mathbb{K}$ is positive;
- *contractive* if $\|f\|\le 1$.

Definition 6.13. An extension $I\rightarrowtail E\twoheadrightarrow Q$ is called *semi-split*, if there exists a completely positive, contractive section $s\colon Q\to E$.

If I,E,Q are graded, then s is required to be graded as well.

Theorem 6.14. *Let $I\rightarrowtail E\twoheadrightarrow Q$ be a semi-split extension of σ-unital C*-algebras. Then:*

- *For any separable A: there is an exact sequence*

$$\begin{array}{ccccc}
\mathrm{KK}(A,I) & \longrightarrow & \mathrm{KK}(A,E) & \longrightarrow & \mathrm{KK}(A,Q)\\
\uparrow & & & & \downarrow\\
\mathrm{KK}(A,\mathrm{S}Q) & \longleftarrow & \mathrm{KK}(A,\mathrm{S}E) & \longleftarrow & \mathrm{KK}(A,\mathrm{S}I).
\end{array}$$

- *If E is separable, then for any σ-unital A: there is an exact sequence*

$$\begin{array}{ccccc}
\mathrm{KK}(I,A) & \longleftarrow & \mathrm{KK}(E,A) & \longleftarrow & \mathrm{KK}(Q,A)\\
\downarrow & & & & \uparrow\\
\mathrm{KK}(\mathrm{S}Q,A) & \longrightarrow & \mathrm{KK}(\mathrm{S}E,A) & \longrightarrow & \mathrm{KK}(\mathrm{S}I,A).
\end{array}$$

The proof in [3] used the Puppe sequence, which we prove first. Recall that the mapping cone $\mathrm{C}_f := \{(a,\varphi)\in A\oplus \mathrm{C}B \mid f(a)=\varphi(1)\}$.

$$\begin{array}{ccc}
\mathrm{C}_f & \xrightarrow{\pi_{\mathrm{C}B}} & \mathrm{C}B\\
{\scriptstyle\pi_A}\downarrow & & \downarrow{\scriptstyle\mathrm{ev}_1}\\
A & \xrightarrow[f]{} & B
\end{array}$$

Theorem 6.15 (Puppe). *There are long exact sequences*

$$\cdots\to \mathrm{KK}(D,\mathrm{S}A)\xrightarrow{(\mathrm{S}f)_*}\mathrm{KK}(D,\mathrm{S}B)\xrightarrow{i_*}\mathrm{KK}(D,\mathrm{C}_f)\xrightarrow{\pi_{A*}}\mathrm{KK}(D,A)\xrightarrow{f_*}\mathrm{KK}(D,B)\to\cdots$$

and

$$\cdots\leftarrow \mathrm{KK}(\mathrm{S}A,D)\xleftarrow{(\mathrm{S}f)^*}\mathrm{KK}(\mathrm{S}B,D)\xleftarrow{i^*}\mathrm{KK}(\mathrm{C}_f,D)\xleftarrow{\pi_A^*}\mathrm{KK}(A,D)\xleftarrow{f^*}\mathrm{KK}(B,D)\leftarrow\cdots,$$

where $\pi_A\colon \mathrm{C}_f\to A$, $\pi(a,\varphi):=a$ and $i\colon \mathrm{S}B\hookrightarrow \mathrm{C}_f$, $i(\varphi):=(0,\varphi)$.

Proof. We only prove the first long exact sequence here. The second is similar but more involved.

- *Exactness at* $\mathrm{KK}(D, A)$. Let $\mathcal{E}_0 = (E_0, \phi_0, F_0) \in \mathbb{E}(D, A)$ satisfy $f_*[\mathcal{E}_0] = [0] \in \mathrm{KK}(D, B)$. Then $f_*\mathcal{E}_0$ is homotopic to 0. Let the homotopy be given by $\tilde{\mathcal{E}} = (\tilde{E}, \tilde{\phi}, \tilde{F}) \in \mathbb{E}(D, \mathrm{I}B)$. Then $(E_0 \oplus \tilde{E}, \phi_0 \oplus \tilde{\phi}, F_0 \oplus \tilde{F}) \in \mathbb{E}(D, \mathrm{C}_f)$, and $\pi_{A*}\tilde{\mathcal{E}} = \mathcal{E}_0$.

  Conversely, let $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(D, \mathrm{C}_f)$. We claim that $f_*\pi_{A*}[\mathcal{E}] = [0]$. By definition of the mapping cone, we have $f \circ \pi_A = \mathrm{ev}_1 \circ \pi_{\mathrm{C}B}$. By functoriality, $f_*\pi_{A*}[\mathcal{E}] = \mathrm{ev}_{1*}(\pi_{\mathrm{C}B*}[\mathcal{E}])$. But $\mathrm{ev}_1 \colon \mathrm{I}B \to B$ is homotopic to $\mathrm{ev}_0 \colon \mathrm{I}B \to B$ via the identity map $\mathrm{I}B \to \mathrm{I}B$, and ev_0 restricted to $\mathrm{C}B$ is the zero map. By homotopy invariance, $\mathrm{ev}_{1*}(\pi_{\mathrm{C}B*}[\mathcal{E}]) = \mathrm{ev}_{0*}(\pi_{\mathrm{C}B*}[\mathcal{E}]) = [0]$.

- *Exactness at* $\mathrm{KK}(D, \mathrm{C}_f)$. Consider the map $\mathrm{C}_f \xrightarrow{\pi_A} A$. Its mapping cone C_{π_A} is

$$\begin{aligned} \mathrm{C}_{\pi_A} &= \{(a, \varphi, \chi) \in A \oplus \mathrm{C}B \oplus \mathrm{C}A \mid f(a) = \varphi(1),\ a = \chi(1)\} \\ &= \{(\varphi, \chi) \in \mathrm{C}B \oplus \mathrm{C}A \mid f(\chi(1)) = \varphi(1)\}. \end{aligned}$$

  Consider the Puppe sequence for $\mathrm{C}_f \xrightarrow{\pi_A} A$. Then the sequence

$$\cdots \longrightarrow \mathrm{KK}(D, \mathrm{C}_{\pi_A}) \xrightarrow{\pi_{\mathrm{C}_f}} \mathrm{KK}(D, \mathrm{C}_f) \xrightarrow{\pi_A} \mathrm{KK}(D, A) \longrightarrow \cdots$$

  is exact at $\mathrm{KK}(D, \mathrm{C}_f)$ by the first part of the proof. Let $\iota \colon \mathrm{S}B \hookrightarrow \mathrm{C}_{\pi_A}$ be the map

$$\iota(\varphi) := (\varphi, 0).$$

  This is a homotopy equivalence: the homotopy inverse is given by $\Phi \colon \mathrm{C}_{\pi_A} \to \mathrm{S}B$,

$$\Phi(\varphi, \chi)(t) := \begin{cases} \varphi(2t) & t \in [0, \frac{1}{2}] \\ f(\chi(2-2t)) & t \in [\frac{1}{2}, 1]. \end{cases}$$

  Notice that $i = \pi_{\mathrm{C}f} \circ \iota$. Therefore, the following diagram commutes:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{KK}(D, \mathrm{C}_{\pi_A}) & \xrightarrow{\pi_{\mathrm{C}_f}} & \mathrm{KK}(D, \mathrm{C}_f) & \xrightarrow{\pi_A} & \mathrm{KK}(D, A) & \longrightarrow \cdots \\ & {\scriptstyle \iota_*}\uparrow & \nearrow_{i_*} & & & & \\ & \mathrm{KK}(D, \mathrm{S}B) & & & & & \end{array}$$

  and by homotopy invariance of KK-theory, ι_* is an isomorphism. Therefore

$$\cdots \longrightarrow \mathrm{KK}(D, \mathrm{S}B) \xrightarrow{i_*} \mathrm{KK}(D, \mathrm{C}_f) \xrightarrow{\pi_A} \mathrm{KK}(D, A) \longrightarrow \cdots$$

  is exact at $\mathrm{KK}(D, \mathrm{C}_f)$.

- *Exactness at* $\mathrm{KK}(D, \mathrm{S}B)$. The proof is essentially the same with the exactness at $\mathrm{KK}(D, \mathrm{C}_f)$: use the Puppe sequence for $\mathrm{S}B \overset{i}{\hookrightarrow} \mathrm{C}_f$ and the homotopy equivalence $\mathrm{S}A \cong \mathrm{C}_i$. Then

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{KK}(D, \mathrm{C}_i) & \xrightarrow{\pi_{\mathrm{S}B*}} & \mathrm{KK}(D, \mathrm{S}B) & \xrightarrow{i_*} & \mathrm{KK}(D, \mathrm{C}_f) & \longrightarrow \cdots \\ & {\scriptstyle \cong}\uparrow & \nearrow_{\mathrm{S}f_*} & & & & \\ & \mathrm{KK}(D, \mathrm{S}A) & & & & & \end{array}$$

  is exact at $\mathrm{KK}(D, \mathrm{S}B)$. This finishes the proof. □

Idea of the proof of Theorem 6.14.

- Let $I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$ be a semi-split extension. Write down the Puppe sequence for $E \overset{q}{\twoheadrightarrow} Q$. Together with the Bott periodicity we obtain the cyclic exact sequence

$$\begin{array}{ccccc} \mathrm{KK}(A, \mathrm{C}_q) & \longrightarrow & \mathrm{KK}(A, E) & \longrightarrow & \mathrm{KK}(A, Q) \\ \uparrow & & & & \downarrow \\ \mathrm{KK}(A, \mathrm{S}Q) & \longleftarrow & \mathrm{KK}(A, \mathrm{S}E) & \longleftarrow & \mathrm{KK}(A, \mathrm{SC}_q). \end{array}$$

- A Lemma:

  Lemma 6.16. *If $I \subseteq E$ is a semi-split ideal. That is, $I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$ is semi-split. Then $I \hookrightarrow \mathrm{C}_q$ is a* KK*-equivalence.*

- Identify C_q with I in the Puppe sequence. □

Corollary 6.17. KK *is split-exact. That is, every split extension of* C^**-algebras induces a* short *exact sequence in* KK*-theory.*

Remark 6.18.

- The above strategy has already been used in my previous talk to prove the long exact sequence in K-theory.
- $I \subseteq E$ is semi-split ideal iff the extension $I \rightarrowtail E \twoheadrightarrow Q$ is an *invertible* extension. Then it gives an element in $\mathrm{Ext}(Q, I)^{-1} \cong \mathrm{KK}_1(Q, I)$. This will be clarified in future talks concerning the extension picture of KK_1.
- Choi and Effros showed in [4] that if A is nuclear, then every ideal of A is semi-split. This is known as the lifting theorem of nuclear C^*-algebras.

April 12 and April 19, 2022

The Kasparov product

Speaker: Bram Mesland (Leiden University)

In this section, we shall always omit the *-homomorphism $\phi\colon A \to \mathbb{B}_B(E)$ in a Kasparov module (E, ϕ, F). While speaking about $(X, F) \in \mathbb{E}(A, B)$, we mean that X is a Hilbert B-module which carries a left A-module structure coming from a *-homomorphism $\phi\colon A \to \mathbb{B}_B(E)$.

We assume all C^*-algebras are separable. Recall that for "nice" properties to hold for $\mathrm{KK}(A, B)$ we shall require that A is separable and B is σ-unital.

7.1 The Kasparov product in the bounded picture

A main property of KK-theory is that there exists an associative bilinear pairing

$$\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \to \mathrm{KK}(A, C)$$

for all separable C^*-algebras A, B and C. This is the Kasparov product (Definition 4.26). But there is a problem: there is no explicit way to write down the Kasparov product of two Kasparov modules, even up to homotopy. This can be solved using the unbounded picture of KK-theory. Before going into that, we shall first investigate the situation in the bounded picture.

Recall the Connes–Skandalis conditions for connections (Definition 4.26):

Theorem 7.1 ([5, Appendix A])**.** *Let $(X, F_1) \in \mathbb{E}(A, B)$ and $(Y, F_2) \in \mathbb{E}(B, C)$. The Kasparov product $(X \otimes_B Y, F) \in \mathbb{E}(A, C)$ is, uniquely up to homotopy, characterised by the following properties:*

1. (Connection condition). *For all $x \in X$: the operator*

$$y \mapsto F(x \otimes_B y) - \gamma(x) \otimes_B F_2 y \qquad \in \mathbb{K}(Y, X \otimes_B Y),$$

where γ is the grading on X.

2. (Positivity condition). *There exists $0 < \kappa < 2$, such that for all $a \in A$:*

$$a^*[F_1 \otimes_B 1, F]a \geq -\kappa a^* a \mod \mathbb{K}(X \otimes_B Y).$$

We aim at constructing the operator F. We would like to think of F as the form $F_1 \otimes_B 1 +$ "$1 \otimes_B F_2$" acting on $X \otimes_B Y$. The naïve idea $F_1 \otimes_B 1 + 1 \otimes_B F_2$ does not work: notice that

$$(1 \otimes_B F_2)(xb \otimes_B y) = xb \otimes_B F_2 y = x \otimes_B bF_2 y$$

but

$$(1 \otimes_B F_2)(x \otimes_B by) = x \otimes_B F_2 by$$

which are not the same unless $[F_2, B] = 0$. This is the degenerate condition (D1).

In order to make sense of "$1 \otimes_B F_2$". We need to apply Kasparov's stablisation theorem.

Let $\hat{\mathbb{Z}} := \mathbb{Z} \setminus 0$ and $\hat{\mathcal{H}} := \ell^2(\hat{\mathbb{Z}})$ equipped with the grading $\deg e_i := \mathrm{sgn}(i)$. This is a graded Hilbert space. For simplicity, assume that B is an ungraded C*-algebra. Define $\hat{\mathcal{H}}_B := \hat{\mathcal{H}} \otimes B$. Kasparov's stablisation theorem (Theorem 3.28) claims that if X is a countably-generated graded Hilbert B-module. Then there exists a graded isometry $V : X \to \hat{\mathcal{H}}_B$. This defines a (tight, normalised) frame of X:

$$x_i := V^*(e_i \otimes 1)$$

satisfying $x = \sum_i x_i \langle x_i, x\rangle$ for all $x \in X$.

Notice that $V \otimes_B 1 : X \otimes_B Y \to \hat{\mathcal{H}}_{B^+} \otimes_B Y = \hat{\mathcal{H}} \otimes B^+ \otimes_B Y \cong \hat{\mathcal{H}} \otimes Y$, and the operator $1 \otimes F_2$ is a well-defined operator on the codomain of $V \otimes_B 1$. This allows us to define the operator

$$\hat{F}_2 := (V^* \otimes_B 1)(1 \otimes F_2)(V \otimes_B 1) : X \otimes_B Y \to X \otimes_B Y.$$
$$\hat{F}_2(x \otimes_B y) = \sum_i \mathrm{sgn}(i) x_i \otimes_B F_2 \langle x_i, x\rangle y.$$

Proposition 7.2. *Define $G := F_1 \otimes_B 1 + \hat{F}_2$. Then G satisfies the connection condition.*

Proof. For simplicity, we only consider the case $x \in X$ with $\deg(x) = 0$. Then

$$x = \sum_{i \geq 0} x_i \langle x_i, x\rangle.$$

Notice that we have

Lemma 7.3. *Let X be a Hilbert B-module. Let Y be a countably-generated Hilbert C-module with a left B-module structure. Fix $x \in X$. The operator*

$$T_x : y \mapsto x \otimes_B y$$

is compact.

Proof of Lemma. Since Y is countably-generated. We may choose a countable frame $\{y_i\}$ of Y. Then $y = \sum_i y_i \langle y_i, y\rangle$ and

$$\begin{aligned} x \otimes_B y &= x \otimes_B \sum_i y_i \langle y_i, y\rangle \\ &= \sum_i x \otimes_B y_i \langle y_i, y\rangle \\ &= \sum_i \Theta_{x \otimes_B y_i, y_i} y. \end{aligned}$$

Hence T_x is the norm limit of a sequence of finite-rank operators. □

We check the connection condition. We have

$$G(x \otimes_B y) - x \otimes_B F_2 y = (F_1 x \otimes y) + (\hat{F}_2(x \otimes_B y) - x \otimes_B F_2 y).$$

By Lemma 7.3, the operator $y \mapsto F_1 x \otimes_B y$ is compact. And

$$\begin{aligned}\hat{F}_2(x \otimes_B y) - x \otimes_B F_2 y &= \sum_{i \geq 0} x_i \otimes_B F_2 \langle x_i, x \rangle y - x \otimes_B F_2 y \\ &= \sum_{i \geq 0} x_i \otimes_B F_2 \langle x_i, x \rangle y - \sum_i x_i \langle x_i, x \rangle \otimes_B F_2 y \\ &= \sum_{i \geq 0} x_i \otimes_B [F_2, \langle x_i, x \rangle] y.\end{aligned}$$

For any finite partial sum, the operator

$$y \mapsto \sum_{i=0}^{N} x_i \otimes_B [F_2, \langle x_i, x \rangle] y$$

is compact by Lemma 7.3. It suffices to check that this operator converges in norm as $N \longrightarrow \infty$. Then it is the norm limit of a sequence of compact operators, hence compact. To this end, notice that

$$\begin{aligned}\sup_{\|y\| \leq 1} \left\| \sum_{i=N+1}^{M} x_i \otimes_B [F_2, \langle x_i, x \rangle y] \right\| &\leq \sup_{\|y\| \leq 1} \|x\| \left\| \begin{pmatrix} [F_2, \langle x_{N+1}, x \rangle] \\ [F_2, \langle x_{N+2}, x \rangle] \\ \vdots \\ [F_2, \langle x_M, x \rangle] \end{pmatrix} \right\| \|y\| \\ &\leq \sup_{\|y\| \leq 1} 2\|x\| \|F_2\| \left\| \begin{pmatrix} \langle x_{N+1}, x \rangle \\ \langle x_{N+2}, x \rangle \\ \vdots \\ \langle x_M, x \rangle \end{pmatrix} \right\| \|y\|.\end{aligned}$$

But since $\sum_i x_i \langle x_i, x \rangle$ converges to x in norm, we have

$$\left\| \begin{pmatrix} \langle x_{N+1}, x \rangle \\ \langle x_{N+2}, x \rangle \\ \vdots \\ \langle x_M, x \rangle \end{pmatrix} \right\| \longrightarrow 0$$

and hence the tail converges to 0 in norm. □

Question Now that we have constructed the operator $G = F_1 \otimes_B 1 + \hat{F}_2$ which satisfies the connection condition. We may ask:

1. Is $(X \otimes_B Y, G)$ a Kasparov module?
2. Does G satisfy the positivity condition?

Answer The answer to both questions is NO! Let us check the conditions:

1. $G = G^*$ is satisfied.
2. $[G, a] = [F_1, a] \otimes_B 1 + [\hat{F}_2, a]$. Although $[F_1, a]$ is compact by (F1), $[F_1, a] \otimes_B 1$ is usually not compact. And we know nothing about $[\hat{F}_2, a]$.
3. $G^2 - 1 = F_1^2 \otimes_B 1 + \hat{F}_2^2 + [F_1 \otimes_B 1, \hat{F}_2] - 1$ and we know nothing about $[F_1 \otimes_B 1, \hat{F}_2]$.
4. Positivity condition: $[F_1 \otimes_B 1, G] = [F_1 \otimes_B 1, F_1 \otimes_B 1] + [F_1 \otimes_B 1, \hat{F}_2]$. We have claimed that $[F_1 \otimes_B 1, F_1 \otimes_B 1]$ is positive as desired, but for the second term $[F_1 \otimes_B 1, \hat{F}_2]$ there is no guarantee.

Kasparov provided a complicated solution to this problem.

Proposition 7.4. *Suppose there are even operators $M, N \in \mathbb{B}_B(X \otimes_B Y)$ with $M + N = 1$, satisfying:*

1. $M(\mathbb{K}(X) \otimes_B 1) \subseteq \mathbb{K}(X \otimes_B Y)$.
2. $N(\hat{F}_2^2 - 1) \in \mathbb{K}(X \otimes_B Y)$, $N[\hat{F}_2, a] \in \mathbb{K}(X \otimes_B Y)$ *and* $N[F_1 \otimes_B 1, \hat{F}_2] \in \mathbb{K}(X \otimes_B Y)$.
3. $[F_1 \otimes_B 1, N] \in \mathbb{K}(X \otimes_B Y)$, $[\hat{F}_2, N] \in \mathbb{K}(X \otimes_B Y)$ *and* $[N, a] \in \mathbb{K}(X \otimes_B Y)$.

Then $M^{1/2}$ and $N^{1/2}$ satisfy 1–3 as well, and

$$(X \otimes_B Y, M^{1/2}(F_1 \otimes_B 1) + N^{1/2}\hat{F}_2) \in \mathbb{E}(A, C)$$

is the Kasparov product of (X, F_1) and (Y, F_2)!

Theorem 7.5 (Kasparov's technical theorem)**.** *Such M and N always exist.*

Remark 7.6. As a consequence, the Kasparov product exists and is unique up to homotopy. This is established through the following process:

1. Use *Kasparov's stablisation theorem* to find $\hat{F}_2$.
2. Use *Kasparov's technical theorem* to make a cycle.
3. Use *Connes–Skandalis' theorem* to prove the existence and uniqueness.

How do we understand the operators M and N? This can be made more clear in the unbounded picture. We can do even better: instead of adding the operators M and N, in the unbounded picture one can usually sum them up directly. In general we may write

$$(X, S) \otimes (Y, T) = (X \otimes_B Y, S \otimes 1 + 1 \otimes_\nabla T)$$

where ∇ is a connection.

7.2 The unbounded picture of KK-theory

Definition 7.7. Let X be a Hilbert B-module. A densely-defined, closed, symmetric operator $D \colon X \supseteq \operatorname{Dom} D \to X$ is *self-adjoint and regular*, if the operator

$$D \pm i \colon X \supseteq \operatorname{Dom} D \to X$$

has dense range.

Remark 7.8. Recall that if D is a self-adjoint operator on a Hilbert space, then it is automatically regular, i.e. $D \pm i$ has dense range. But this does not hold for operators on Hilbert C*-modules.

Then the operator $(D \pm i)^{-1}$ is contractive and densely defined, hence extends to a bounded adjointable operator on X. We also have $\operatorname{Dom} D = \operatorname{Ran}(D \pm i)^{-1}$.

Definition 7.9. Let A and B be separable C*-algebras. An unbounded Kasparov (A, B)-module is a pair (X, D) with:

- X is a Hilbert B-module, which also carries a left A-module structure.
- $D \colon X \supseteq \operatorname{Dom} D \to X$ is self-adjoint, regular operator, satisfying
  - $a(D \pm i)^{-1} \in \mathbb{K}_B(X)$.
  - The subset (which is a *-subalgebra of A)

$$\operatorname{Lip}(A) := \{a \in A \mid a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D \text{ and } [D, a] \in \mathbb{B}_B(X)\}$$

is norm-dense in A.

We denote the set of unbounded Kasparov (A,B)-modules by $\Psi(A,B)$.

Example 7.10.

1. $(L^2(\mathbb{S}^1), i\frac{\mathrm{d}}{\mathrm{d}x}) \in \Psi(\mathrm{C}(\mathbb{S}^1), \mathbb{C})$. In particular: notice that the dense subalgebra

$$\left\{ f \in \mathrm{C}(\mathbb{S}^1) \,\middle|\, f\colon \operatorname{Dom} i\frac{\mathrm{d}}{\mathrm{d}x} \to \operatorname{Dom} i\frac{\mathrm{d}}{\mathrm{d}x},\ [D,f] \text{ is bounded} \right\}$$

is just the algebra of Lipschitz functions $\mathrm{Lip}(\mathbb{S}^1)$ on $\mathbb{S}^1$.

2. Let M be a closed smooth manifold. Then $(L^2(M, \Lambda^*\mathrm{T}^*M), \mathrm{d}+\mathrm{d}^*) \in \Psi(\mathrm{C}(M), \mathbb{C})$.
3. $(L^2(\mathbb{R}), i\frac{\mathrm{d}}{\mathrm{d}x}) \in \Psi(\mathrm{C}_0(\mathbb{R}), \mathbb{C})$.
4. $(\mathrm{C}_0(\mathbb{R}), x) \in \Psi(\mathbb{C}, \mathrm{C}_0(\mathbb{R}))$. Note that $(x+i)^{-1} \in \mathrm{C}_0(\mathbb{R})$.

Lemma 7.11. *If D is self-adjoint and regular. Then $(1+D^2)^{-1/2} \in \mathbb{B}_B(X)$ and* $\operatorname{Dom} D = \operatorname{Ran}(1+D^2)^{-1/2}$.

Corollary 7.12. *By the closed graph theorem: $D(1+D^2)^{-1/2}$ is everywhere defined and has closed range. Hence $D(1+D^2)^{-1/2} \in \mathbb{B}_B(X)$.*

We call $F_D \coloneqq D(1+D^2)^{-1/2}$ the bounded transform *of D.*

Theorem 7.13 ([2]). *Let $(X,D) \in \Psi(A,B)$. Then:*

1. *$(X, F_D) \in \mathbb{E}(A,B)$.*
2. *The map*

$$\Psi(A,B) \to \mathrm{KK}(A,B), \qquad (X,D) \mapsto [X, F_D]$$

is surjective.

Remark 7.14. The second part of the theorem does not state that every bounded Kasparov module in $\mathbb{E}(A,B)$ lifts to an unbounded module in $\Psi(A,B)$: this is not true. The lift is only possible up to homotopy.

Proof. We only sketch the proof of 1. For simplicity, assume that A is unital. We need to check that $F_D = D(1+D^2)^{-1/2}$ satisfies (F1)–(F3). $F_D^* = F_D$ because $D = D^*$. For $F_D^2 - 1$, we have

$$\begin{aligned} F_D^2 - 1 = D(1+D^2)^{-1/2}D(1+D^2)^{-1/2} - 1 = D^2(1+D^2)^{-1} - 1 \\ = -(1+D^2)^{-1} = -((D+i)(D-i))^{-1} = -(D-i)^{-1}(D+i)^{-1} \in \mathbb{K}_B(X). \end{aligned}$$

(To convince you all these are legal: it is straightforward that $(1+D^2)^{-1/2}D \subseteq D(1+D^2)^{-1/2}$. Now check the domain. Recall that $(1+D^2)^{-1/2}$ maps to $\operatorname{Dom}(D^2) \subseteq \operatorname{Dom}(D)$. Restricted to this domain $(1+D^2)^{-1/2}D = D(1+D^2)^{-1/2}$. So the second equality holds. In the second line: we have that $D^2+1, D \pm i$ are surjective onto X with bounded inverses. In particular: $D^2 + 1 = (D+i)(D-i)$ on $\operatorname{Dom}(D^2)$. Then their inverse must be equal.)

The most non-trivial part is to show that $[F_D, a]$ is compact. We have

$$[F_D, a] = [D(1+D^2)^{-1/2}, a] = [D,a](1+D^2)^{-1/2} + D[(1+D^2)^{-1/2}, a].$$

The first term

$$[D,a](1+D^2)^{-1/2} = [D,a](D-i)^{-1/2}(D+i)^{-1/2}$$

is compact, because $[D,a]$ is bounded, and $D \pm i$ are compact.

For the second term: we need to use

$$T^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(T+\lambda^2)^{-1}\,\mathrm{d}\lambda. \tag{3}$$

and

$$[a^{-1}, b] = -a^{-1}[a,b]a^{-1}. \tag{4}$$

Now

$$\begin{aligned}
D[(1+D^2)^{-1/2},a] &\overset{(3)}{=} \frac{D}{\pi}\int_0^\infty \lambda^{-1/2}[(1+\lambda^2+D^2)^{-1},a]\,\mathrm{d}\lambda \\
&\overset{(4)}{=} -\frac{D}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda^2+D^2)^{-1}[1+\lambda^2+D^2,a](1+\lambda^2+D^2)^{-1}\,\mathrm{d}\lambda \\
&= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+\lambda^2+D^2)^{-1}[D^2,a](1+\lambda^2+D^2)^{-1}\,\mathrm{d}\lambda \\
&= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+\lambda^2+D^2)^{-1}D[D,a](1+\lambda^2+D^2)^{-1}\,\mathrm{d}\lambda \\
&\quad -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+\lambda^2+D^2)^{-1}[D,a]D(1+\lambda^2+D^2)^{-1}\,\mathrm{d}\lambda.
\end{aligned}$$

Use the fact that $[D,a]$ is bounded and $D(1+\lambda^2+D^2)^{-1}D$ is contractive (by functional calculus):

$$\begin{aligned}
\|D[(1+D^2)^{-1/2},a]\| &\le \frac{2}{\pi}\|[D,a]\|\int_0^\infty \lambda^{-1/2}(1+\lambda^2+D^2)^{-1}\,\mathrm{d}\lambda \\
&\le \frac{2}{\pi}\|[D,a]\|\int_0^\infty \lambda^{-1/2}(1+\lambda^2)^{-1}\,\mathrm{d}\lambda
\end{aligned}$$

The integral

$$\int_0^\infty \lambda^{-1/2}(1+\lambda^2)^{-1}\,\mathrm{d}\lambda$$

converges absolutely. (Notice that $\lambda^{-1/2}(1+\lambda^2)^{-1}\sim\lambda^{-1/2}$ as $\lambda\longrightarrow 0$, and $\lambda^{-1/2}(1+\lambda^2)^{-1}\sim\lambda^{-3/2}$ as $\lambda\longrightarrow\infty$). Therefore $\|D[(1+D^2)^{-1/2},a]\|$ is bounded and we conclude that $[F_D,a]$ is bounded. □

A big advantage of the unbounded picture is that one can simply sum up the unbounded operators to obtain the Kasparov product in ideal situations.

7.3 The Kasparov product in the unbounded picture

We have the following Connes–Skandalis type theorem of Kucerovsky.

Theorem 7.15 ([16]). *Let $(X,S)\in\Psi(A,B)$ and $(Y,T)\in\Psi(B,C)$. If $(X\otimes_B Y,D)\in\Psi(A,C)$ satisfies the following conditions:*

1. (Connection condition) *For all x in a dense subspace of X: the operator*

$$y\mapsto D(x\otimes y)-\gamma(x)\otimes Ty$$

   extends to a bounded (hence adjointable) operator in $\mathbb{B}_C(Y,X\otimes_B Y)$.
2. (Domain condition) $\operatorname{Dom} D\subseteq\operatorname{Dom}(S\otimes_B 1)$.
3. (Positivity and boundedness condition) *There exists a real number κ such that for all $\xi\in\operatorname{Dom} D$:*

$$\langle D\xi+(S\otimes_B 1)\xi\rangle+\langle(S\otimes_B 1)\xi,D\xi\rangle\ge\kappa\langle\xi,\xi\rangle.$$

Then $(X\otimes_B Y,D)$ represents the Kasparov product of (X,S) and (Y,T).

Remark 7.16.

1. The domain condition indicates that we should think of D as an operator

$$D=S\otimes_B 1+\hat{T},$$

   hence $\operatorname{Dom} D=\operatorname{Dom} S\otimes_B 1\cap\operatorname{Dom}\hat{T}\subseteq\operatorname{Dom} S\otimes_B 1$.
2. The positivity and boundedness condition is, more or less, a rephrasal of the positivity condition in the unbounded picture. But we need to care about the domain issue: we do not yet know whether D and S are composable, so the graded commutator may not make sense.

7.3.1 The exterior Kasparov product

Before constructing the interior Kasparov product of unbounded modules, it is beneficial to look at the *exterior product.* This is an associative bilinear map

$$\mathrm{KK}(A, B) \times \mathrm{KK}(C, D) \to \mathrm{KK}(A \otimes C, B \otimes D).$$

In the bounded picture. Given $(X, F_1) \in \mathbb{E}(A, B)$ and $(Y, F_2) \in \mathbb{E}(C, D)$. The exterior product is represented by

$$(X \otimes Y, M^{1/2}(F_1 \otimes 1) + N^{1/2}(1 \otimes F_2)) \in \mathbb{E}(A \otimes C, B \otimes D),$$

where the operators M and N appear again. But they no longer present if we work with unbounded modules.

Theorem 7.17 (Baaj–Julg)**.** *The exterior Kasparov product of $(X, S) \in \Psi(A, B)$ and $(Y, T) \in \Psi(C, D)$ is represented by*

$$(X \otimes Y, S \otimes 1 + 1 \otimes T) \in \Psi(A \otimes C, B \otimes D).$$

Remark 7.18. Notice that $S \otimes 1$ and $1 \otimes T$ anticommute. In fact:

$$\begin{aligned}((S \otimes 1)(1 \otimes T) + (1 \otimes T)(S \otimes 1))(x \otimes y) &= (S \otimes 1)\gamma(x) \otimes Ty + (1 \otimes T)(Sx \otimes y)\\ &= S\gamma(x) \otimes Ty + \gamma(Sx) \otimes Ty\\ &= S\gamma(x) \otimes Ty - S\gamma(x) \otimes Ty = 0.\end{aligned}$$

Now we can answer the question: what are the operators M and N in the bounded picture?

Proposition 7.19. *Suppose s and t are self-adjoint and regular operators on a Hilbert B-module X, such that:*

- $C := \mathrm{Dom}\, ts \cap \mathrm{Dom}\, st$ *is a common core for both s and t.*
- $st + ts = 0$ *on C.*

Then $(s + t)^2 = s^2 + t^2$ on $\mathrm{Dom}\, s^2 \cap \mathrm{Dom}\, t^2$.

Set $s := S \otimes 1$ and $t := 1 \otimes T$. Then they satisfy the conditions in the previous proposition. Consider the bounded transform of the unbounded operator $s + t$. We have

$$\begin{aligned}(s + t)(1 + (s + t)^2)^{-1/2} &= s(1 + s^2 + t^2)^{-1/2} + t(1 + s^2 + t^2)^{-1/2}\\ &= s(\frac{1}{2} + s^2)^{-1/2} \boxed{(\frac{1}{2} + s^2)^{1/2}(1 + s^2 + t^2)^{-1/2}} \longrightarrow M\\ &\quad + t(\frac{1}{2} + t^2)^{-1/2} \boxed{(\frac{1}{2} + t^2)^{1/2}(1 + s^2 + t^2)^{-1/2}} \longrightarrow N\end{aligned}$$

The operator $s(\frac{1}{2} + s^2)^{-1/2}$ is, up to a rescaling, the bounded transform of s: notice that

$$D(\lambda^2 + D)^{-1/2} = \frac{D}{\lambda}\left(1 + \left(\frac{D}{\lambda}\right)^2\right)^{-1/2}.$$

So a scaling provides a homotopy of unbounded modules. The scaling homotopy is not true in the bounded picture.

7.3.2 Connections

Back to the interior Kasparov products. A similar problem occurs as in the bounded picture: we need to make sense of "$1 \otimes_B T$" acting on $X \otimes_B Y$. We need to define connections in the unbounded context.

For simplicity, let B be ungraded.

Definition 7.20. Let $(Y, T) \in \Psi(B, C)$. We choose a *-subalgebra $\mathcal{B} \subseteq \operatorname{Lip}(B)$ which is dense in the norm of B: this is part of the data. We unpack such data as a triple $(\mathcal{B}, Y, T)$.

The noncommutative differential 1-form for $(\mathcal{B}, Y, T)$ is

$$\Omega^1_T(\mathcal{B}) := \overline{\operatorname{span}}\{b[T, b'] \mid b, b' \in \mathcal{B}\} \subseteq \mathbb{B}_C(Y).$$

The closure is with respect to the norm topology of $\mathbb{B}_C(Y)$.

Remark 7.21. $\Omega^1_T(\mathcal{B})$ is a $\mathcal{B}, \mathcal{B}$-bimodule. The left module structure is obvious. The right module structure is obtained by enforcing the Leibniz rule:

$$[T, b]c := [T, bc] - b[T, c].$$

Definition 7.22. Let X be a Hilbert B-module. A Hermitian $(\mathcal{B}, Y, T)$-connection on X is a densely defined linear map

$$\nabla \colon X \supseteq \mathcal{X} \to X \otimes^{\mathrm{h}}_{\mathcal{B}} \Omega^1_T(\mathcal{B}) \subseteq X \otimes^{\mathrm{h}}_B \mathbb{B}_C(Y),$$

($\mathcal{X} := \operatorname{Dom} \nabla$. $\otimes^{\mathrm{h}}$ denotes the *Haagerup tensor product*), such that

- $\nabla(xb) = \nabla(x)b + \gamma(x) \otimes_B [T, b]$ for $b \in \mathcal{B}$.
- $\langle x_1, \nabla x_2\rangle - \langle \nabla x_1, x_2\rangle = [T, \langle x_1, x_2\rangle]$.

Remark 7.23. Why Haagerup tensor product?

- The Haagerup tensor product is characterised by the property that the multiplication map
$$B \otimes^{\mathrm{h}}_B B \xrightarrow{\text{mult}} B$$
is continuous for any C*-algebra B.
- More generally: the Haagerup tensor product is characterised by the following property: given any C*-algebra B and Hilbert B-module X, the multiplication map
$$X \otimes^{\mathrm{h}}_B B \xrightarrow{\text{mult}} X$$
is continuous.
- We can say even more. Given a Hilbert B-module X and a Hilbert C-module Y which carries a left B-module structure. Then there is a completely bounded isomorphism
$$X \otimes_B Y \cong X \otimes^{\mathrm{h}}_B Y.$$
Here we should view both sides as operator modules.

Remark 7.24. A connection always exists.

Definition 7.25. Given $(\mathcal{B}, Y, T)$ and a densely defined $(\mathcal{B}, Y, T)$-connection ∇ on X. Define the operator

$$1 \otimes_\nabla T \colon X \otimes_B Y \supseteq \mathcal{X} \otimes^{\mathrm{alg}}_{\mathcal{B}} \operatorname{Dom} T \to X \otimes_B Y$$
$$(1 \otimes_\nabla T)(x \otimes y) := \gamma(x) \otimes_B Ty + \nabla(x)y.$$

We have the satisfying result:

Proposition 7.26. *$D := S \otimes 1 + 1 \otimes_\nabla T$ is well-defined and satisfies Kucerovsky's connection condition.*

Proof. We have

$$\begin{aligned} 1 \otimes_\nabla T(xb \otimes_B y) &= \gamma(xb) \otimes_B Ty + \nabla(xb)y \\ &= \gamma(x) \otimes_B bTy + \nabla(x)by + \gamma(x) \otimes_B [T, b]y \\ &= \gamma(x) \otimes_B Tby + \nabla(x)by \\ &= 1 \otimes_\nabla T(x \otimes_B by). \end{aligned}$$

So the operator $1 \otimes_\nabla T$ is well-defined. Now

$$(S \otimes 1 + 1 \otimes_\nabla T)(x \otimes_B y) - \gamma(x) \otimes Ty = Sx \otimes_B y + \nabla(x)y.$$

We have proven that $y \mapsto Sx \otimes_B y$ is compact, hence bounded. But $y \mapsto \nabla(x)y$ is obviously bounded because $\nabla(x) \in X \otimes^{\mathrm{h}}_B \mathbb{B}_C(Y)$. ◻

7.3.3 The interior Kasparov product

To finalise the construction of the Kasparov product, we need that operator $D := S \otimes 1 + 1 \otimes_\nabla T$ satisfies:

(1) D is self-adjoint and regular.

(2) D has compact resolvent. i.e. $a(D \pm i)^{-1} \in \mathbb{K}$.

(3) $[D, a] \in \mathbb{B}$ for $a \in \mathrm{Lip}(A)$, or a dense *-subalgebra $\mathcal{A} \subseteq A$ contained in $\mathrm{Lip}(A)$.

When are these condition satisfied?

(2) is automatically true by some long computation.

(1) is usually not guaranteed. But this is closely related to the graded commutator $[S \otimes 1, 1 \otimes_\nabla T]$ and hence the *Positivity and Boundedness condition.* If it is *relatively bounded* by both $S \otimes 1$ and $1 \otimes_\nabla T$ then this is true.

(3) is quite independent and indicates that the connection should be compatible with A in a suitable sense.

Eventually, we have

Theorem 7.27 (informally). *Let $(X, S) \in \Psi(A, B)$ and $(Y, T) \in \Psi(B, C)$. Pick a dense *-subalgebra $\mathcal{B} \subseteq B$ satisfying the "Lipschitz" conditions. Let ∇ be a $(\mathcal{B}, Y, T)$-connection on X. If:*

- *$S \otimes_B 1$ and $1 \otimes_\nabla T$ have "small anticommutator".*
- *$[1 \otimes_\nabla T, a]$ extends to a bounded adjointable operator for all $a \in \mathrm{Lip}(A)$.*

Then $(X \otimes_B Y, S \otimes_B 1 + 1 \otimes_\nabla T)$ represents the Kasparov product.

Example 7.28. Given $(\mathrm{C}_c^\infty(\mathbb{R}), \mathrm{C}_0(\mathbb{R}), x) \in \Psi(\mathbb{C}, \mathrm{C}_0(\mathbb{R}))$ and $(\mathrm{C}_c^\infty, L^2(\mathbb{R}), i\frac{\mathrm{d}}{\mathrm{d}x}) \in \Psi(\mathrm{C}_0(\mathbb{R}), \mathbb{C})$. Their Kasparov product is represented by

$$\left(\mathrm{C}_c^\infty(\mathbb{R}), L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \begin{pmatrix} 0 & x + \frac{\mathrm{d}}{\mathrm{d}x} \\ x - \frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix}\right) \in \Psi(\mathbb{C}, \mathbb{C}).$$

May 3 and May 10, 2022

Extension of C*-algebras and KK-theory

Speaker: Georg Huppertz (Radboud University Nijmegen)

By an *extension*, or more concretely *an extension of C by A*, we shall always mean a short exact sequence of C*-algebras $A \rightarrowtail B \twoheadrightarrow C$.

Let A be a C*-algebra. We write $\mathcal{M}(A)$ for the *multiplier algebra* of A, and $\mathcal{Q}(A) := \mathcal{M}(A)/A$ is the corona algebra. We have the following extension of C*-algebras:

$$A \rightarrowtail \mathcal{M}(A) \twoheadrightarrow \mathcal{Q}(A).$$

8.1 Busby invariant

Recall the multiplier algebras can be realised as double centralisers:

Definition 8.1. Let A be a C*-algebra. A double centraliser of A is a pair (L, R) where $L, R \in \mathbb{B}(A)$ are bounded linear maps, satisfying

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad R(a)b = aL(b), \qquad \text{for all } a, b \in A.$$

The C*-algebra $\mathcal{M}(A)$ consists of double centralisers of A as elements, with multiplication, involution and norm:

$$(L_1, R_1) \cdot (L_2, R_2) := (L_1 L_2, R_2 R_1), \qquad (L, R)^* := (R^*, L^*), \qquad \|(L, R)\| := \|L\| = \|R\|.$$

Then $A \subseteq \mathcal{M}(A)$ via $a \mapsto (L_a, R_a)$ where L_a and R_a denote left and right multiplication by a. If A is unital, then $A \cong \mathcal{M}(A)$.

Lemma 8.2 (Universal property of the multiplier algebra)**.** *Let $I \subseteq A$ be an ideal. Then there exists a unique *-homomorphism $A \xrightarrow{\phi} \mathcal{M}(I)$, such that the following diagram commutes:*

$$\begin{array}{ccc} I & \hookrightarrow & A \\ & \searrow & \downarrow{\scriptstyle \phi} \\ & & \mathcal{M}(I). \end{array}$$

Proof. Since I is an ideal in A, the map $\phi(a) = (L_a, R_a)$ defines a multiplier of I. It suffices to show the uniqueness. If there is another $\psi\colon A \to \mathcal{M}(I)$ making the diagram commute, then

$$\psi(a)i = \psi(a)\psi(i) = \psi(ai) = ai$$

for all $a \in A$, $i \in I$. Therefore $\psi(a) = \phi(a)$ for all $a \in A$. □

Theorem 8.3. *Let $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ be an extension. There exists a unique *-homomorphism $\sigma\colon B \to \mathcal{M}(A)$ and a unique *-homomorphism $\tau\colon C \to \mathcal{Q}(A)$ such that the diagram*

$$\begin{array}{ccccc} A & \overset{\alpha}{\rightarrowtail} & B & \overset{\beta}{\twoheadrightarrow} & C \\ \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ A & \rightarrowtail & \mathcal{M}(A) & \underset{\pi}{\twoheadrightarrow} & \mathcal{Q}(A) \end{array}$$

commutes. Here $\mathcal{Q}(A)$ is the corona algebra $\mathcal{M}(A)/A$.

Proof. The existence and uniqueness of $\sigma\colon B \to \mathcal{M}(A)$ is by universal property. Since β is surjective, for any $c \in C$, pick $b \in \beta^{-1}(c)$. Define

$$\tau(c) := \pi \circ \sigma(b).$$

It is easy to show that τ is a well-defined *-homomorphism. (The surjectivity of β predetermines τ). □

Definition 8.4. The *-homomorphism $\tau\colon C \to \mathcal{Q}(A)$ defined as above is called the *Busby invariant* of the extension $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$.

Theorem 8.5. *Every *-homomorphism $C \xrightarrow{\tau} \mathcal{Q}(A)$ is a Busby invariant of some extension.*

Proof. Define B_τ as in the pullback diagram

$$\begin{array}{ccc} B_\tau & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{M}(A) & \overset{\pi}{\twoheadrightarrow} & \mathcal{Q}(A). \end{array}$$

Then $A \rightarrowtail B_\tau \twoheadrightarrow C$ has Busby invariant τ. □

Definition 8.6. Let $A \overset{\alpha_1}{\rightarrowtail} B_1 \overset{\beta_1}{\twoheadrightarrow} C$ and $A \overset{\alpha_2}{\rightarrowtail} B_2 \overset{\beta_2}{\twoheadrightarrow} C$ be extensions. We say they are *isomorphic*, if there exists a *-homomorphism $B_1 \xrightarrow{\phi} B_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C \\ \| & & \downarrow{\scriptstyle \phi} & & \| \\ A & \xrightarrow{\alpha_2} & B_2 & \overset{\beta_2}{\twoheadrightarrow} & C. \end{array}$$

By Five Lemma, such ϕ must be an isomorphism of C*-algebras.

Theorem 8.7. *There is a 1-1 correspondence between:*

- **-homomorphisms $C \to Q(A)$.*
- *Isomorphism classes of extensions of C by A.*

Proof. We have seen that every *-homomorphism $C \to Q(A)$ gives rise to an extension. We claim that this correspondence is injective up to isomorphism. Let $A \rightarrowtail B \twoheadrightarrow C$ be an extension with Busby invariant $C \xrightarrow{\tau} Q(A)$. Then the *-homomorphism $B \xrightarrow{(\sigma,\beta)} B_\tau$ defines an isomorphism between the extensions $A \rightarrowtail B \twoheadrightarrow C$ and $A \rightarrowtail B_\tau \twoheadrightarrow C$. □

Definition 8.8.

- An extension $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ is *split* if there exists a *-homomorphism $C \xrightarrow{\gamma} B$ such that $\beta \circ \gamma = \mathrm{id}_C$.
- An extension $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ is *orthogonal* if it is isomorphic to the extension $A \rightarrowtail A \oplus C \twoheadrightarrow C$.

Example 8.9. Let A be a non-unital C*-algebra. Then $A \rightarrowtail A^+ \twoheadrightarrow \mathbb{C}$ splits, but is not orthogonal in general: if $A \rightarrowtail B \twoheadrightarrow C$ is orthogonal, then the image of C under the splitting is an ideal in B. But $\mathbb{C}$ is not an ideal of A^+ in general.

Theorem 8.10. *An extension $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ splits iff there exists a *-homomorphism $\eta\colon C \to \mathcal{M}(A)$ such that $\tau = \pi \circ \eta$, where $\pi\colon \mathcal{M}(A) \to Q(A)$.*

Proof. Suppose $A \rightarrowtail B \twoheadrightarrow C$ splits by γ. Define $\eta := C \xrightarrow{\gamma} B \xrightarrow{\sigma} \mathcal{M}(A)$. Then η satisfies $\pi\circ\eta = \pi\circ\sigma\circ\gamma = \tau\circ\beta\gamma = \tau$. Conversely, the *-homomorphism $\eta\colon C \to \mathcal{M}(A)$ defines a split $C \xrightarrow{(\eta,\mathrm{id})} B_\tau$ for the extension $A \rightarrowtail B_\tau \twoheadrightarrow C$. This extension is isomorphic to $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$, hence the latter splits too. □

Definition 8.11. Let $\tau_1, \tau_2\colon C \rightrightarrows Q(A)$ be two Busby invariants. We say they are *unitarily equivalent*, if there exists a unitary multiplier $u \in \mathcal{M}(A)$ such that

$$\tau_2(c) = \pi(u)\tau_1(c)\pi(u^*) \qquad \text{for all } c \in C.$$

Remark 8.12.

- The Busby invariant of an orthogonal extension is the zero map.
- Orthogonal extensions can only be unitarily equivalent to orthogonal extensions.
- Split extensions can only be unitarily equivalent to split extensions.

8.2 The Ext group

Definition 8.13. We use the following notations:

- $\mathfrak{Ext}(A,B)$: the set of isomorphism classes of extensions of A by $\mathbb{K} \otimes B$.
- $\mathfrak{Dxt}(A,B)$: the set of isomorphism classes of split extensions of A by $\mathbb{K} \otimes B$.
- $\overline{\mathfrak{Ext}}(A,B)$: the set of unitary equivalence classes of extensions of A by $\mathbb{K} \otimes B$.
- $\overline{\mathfrak{Dxt}}(A,B)$: the set of unitary equivalence classes of split extensions of A by $\mathbb{K} \otimes B$.

Recall that $\mathbb{K}_B(\mathcal{H}_B) \cong \mathbb{K} \otimes B$ and $\mathbb{B}_B(\mathcal{H}_B) \cong \mathcal{M}(\mathbb{K} \otimes B)$. Pick a specific isomorphism $\mathcal{H}_B \xrightarrow{\cong} \mathcal{H}_B \oplus \mathcal{H}_B$. This induces isomorphisms $\mathbb{M}_2(\mathbb{B}_B(\mathcal{H}_B)) \xrightarrow{\cong} \mathbb{B}_B(\mathcal{H}_B)$ and $\mathbb{M}_2(Q(\mathbb{K} \otimes B)) \cong Q(\mathbb{K} \otimes B)$. This allows us to define additions on the sets defined above by passing to Busby invariants.

Definition 8.14. Let $\phi_1, \phi_2\colon A \to Q(\mathbb{K} \otimes B)$ be Busby invariants, so they represent elements in $\mathfrak{Ext}(A,B)$. Define $\phi_1 \oplus \phi_2\colon A \to Q(\mathbb{K} \otimes B)$ via

$$(\phi_1 \oplus \phi_2)(a) := \begin{pmatrix} \phi_1(a) & \\ & \phi_2(a) \end{pmatrix} \in \mathbb{M}_2 Q(\mathbb{K} \otimes B) \cong Q(\mathbb{K} \otimes B).$$

$\mathfrak{Ext}(A,B)$ becomes a semigroup under the addition.

Remark 8.15.
- Since the sum of split extensions is again a split extension, the addition descends to $\mathfrak{Dxt}(A, B)$.
- If $\phi_1 \sim \phi_1'$ and $\phi_2 \sim \phi_2'$ are two pairs of unitarily equivalent Busby invariants in $\overline{\mathfrak{Ext}}(A, B)$, then $\phi_1 \oplus \phi_2 \sim \phi_1' \oplus \phi_2'$ are unitarily equivalent. So the addition descends to $\overline{\mathfrak{Ext}}(A, B)$ and $\overline{\mathfrak{Dxt}}(A, B)$.
- On $\overline{\mathfrak{Ext}}(A, B)$, the addition is abelian:

$$\begin{pmatrix} \phi_2(a) & \\ & \phi_1(a) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(a) & \\ & \phi_2(a) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- On $\overline{\mathfrak{Ext}}(A, B)$, the addition does not depend on the choice of the isomorphism $\mathcal{H}_B \xrightarrow{\cong} \mathcal{H}_B \oplus \mathcal{H}_B$.

Definition 8.16.

$$\mathrm{Ext}(A, B) := \overline{\mathfrak{Ext}}(A, B)/\overline{\mathfrak{Dxt}}(A, B).$$

Proposition 8.17. $\mathrm{Ext}(A, B)$ *is a group.*

Definition 8.18. Let A and B be separable C*-algebras. By an (A, B)*-pair* we shalll mean a pair (ϕ, P) where $\phi\colon A \to \mathcal{M}(B \otimes \mathbb{K})$ is a *-homomorphism, and $P \in \mathcal{M}(B \otimes \mathbb{K})$, such that:

$$\phi(a)P - P\phi(a) \in B \otimes \mathbb{K}, \quad (P^2 - P)\phi(a) \in B \otimes \mathbb{K}, \quad (P - P^*)\phi(a) \in B \otimes \mathbb{K}, \qquad \text{for all } a \in A.$$

We say an (A, B)-pair (ϕ, P) is *degenrate*, if all above are equal to 0. Denote by $\mathcal{E}^1(A, B)$ the class of all (A, B)-pairs and $\mathcal{D}^1(A, B)$ the class of all degenerate (A, B)-pairs.

Example 8.19. Some examples of an (A, B)-pair (ϕ, P):

1. $\phi\colon A \to \mathcal{M}(B \otimes \mathbb{K})$ is any *-homomorphism and $P \in B \otimes \mathbb{K}$.
2. $\phi\colon A \to \mathcal{M}(B \otimes \mathbb{K})$ is any *-homomorphism and $P = \mathrm{id}$. Then (ϕ, P) is degenerate.
3. $\phi = 0$ and $P \in \mathcal{M}(B \otimes \mathbb{K})$. Then $(0, P)$ is degenerate.

We have operations on $\mathcal{E}^1(A, B)$:

Definition 8.20.
- Let (ϕ_1, P_1) and (ϕ_2, P_2) be (A, B)-pairs. Their sum is defined as

$$(\phi_1, P_1) + (\phi_2, P_2) := (\phi_1 \oplus \phi_2, P_1 \oplus P_2)$$

with the identification $\mathcal{M}(B \otimes \mathbb{K}) \cong \mathbb{M}_2(\mathcal{M}(B \otimes \mathbb{K}))$.
- Let (ϕ_1, P_1) and (ϕ_2, P_2) be (A, B)-pairs. We say they are *unitarily equivalent*, if there exists a unitary multiplier $u \in \mathcal{M}(B \otimes \mathbb{K})$ such that

$$\phi_2(a) = u\phi_1(a)u^*, \quad P_2 = uP_1u^*, \qquad \text{for all } a \in A.$$

Proposition 8.21.
- *Degenerate pairs can only be unitarily equivalent to degenerate pairs.*
- *Addition of pairs descends to their unitary equivalence classes.*

Definition 8.22. We say two (A, B)-pairs (ϕ_1, P_1) and (ϕ_2, P_2) are *homological*[2], if

$$P_1\phi_1(a) - P_2\phi_2(a) \in B \otimes \mathbb{K} \quad \text{for all } a \in A.$$

Remark 8.23. Degenerate pairs can be homological to non-degenerate ones. For example, 1 and 3 in Example 8.19 are homological. But 3 is degenerate while 1 is not.

[2]The term “homological” here was used by Kasparov. I am, however, not a big fan of this name, for some reasons. A reason is that there are too many other properties that deserve this name; another is grammatically: it seems to me that “homological” should be a property of *one* object, not a relation of a pair of objects.

Definition 8.24. Define

$$\overline{\mathcal{E}^1(A,B)} := \mathcal{E}^1(A,B) \Big/ \text{unitary equivalence and homology.}$$

$$\overline{\mathcal{D}^1(A,B)} := \text{Classes in } \mathcal{E}^1(A,B) \text{ such that there is a degenerate representative.}$$

$$\mathrm{E}^1(A,B) := \overline{\mathcal{E}^1(A,B)} \Big/ \overline{\mathcal{D}^1(A,B)}.$$

Theorem 8.25. $\mathrm{E}^1(A,B)$ *is a semigroup. If A is separable and nuclear, B is σ-unital, then*

$$\mathrm{E}^1(A,B) \to \mathrm{Ext}(A,B), \qquad [\phi, P] \mapsto [P\phi]$$

is a semigroup isomorphism. Here $[P\phi]$ is the Busby invariant defined by the map $a \mapsto P\phi(a)$.

Proof. We first show that this map is well-defined. Since

$$\begin{aligned} P\phi(ab) \equiv P^2\phi(ab) \equiv P\phi(a)P\phi(b) \mod B \otimes \mathbb{K}, \\ (P\phi(a))^* \equiv \phi(a^*)P^* \equiv P\phi(a^*) \mod B \otimes \mathbb{K}. \end{aligned}$$

So $P\phi$ defines a *-homomorphism $A \to \mathcal{Q}(B \otimes \mathbb{K})$. Hence it is the Busby invariant of some extension.

Clearly the addition and homology are preserved. Let $u \in \mathcal{M}(B \otimes \mathbb{K})$ be a unitary multiplier. Then $uP\phi(a)u^*$ is unitarily equivalent to $P\phi(a)$ modulo $B \otimes \mathbb{K}$. So the unitary equivalence is also preserved. Now we check that degenerate pairs are sent to split extensions. This is because a degenerate pair defines a *-homomorphism $A \to \mathcal{M}(B \otimes \mathbb{K})$. Hence it defines a split extension.

Therefore the map $\mathrm{E}^1(A,B) \to \mathrm{Ext}(A,B)$, $[\phi, P] \mapsto [P\phi]$ is indeed well-defined. We claim that it is injective. If (ϕ_1, P_1) and (ϕ_2, P_2) defines the same element in $\mathrm{Ext}(A,B)$, that is, there exists unitary multiplier $u \in \mathcal{M}(B \otimes \mathbb{K})$ and $\psi_1, \psi_2 \in \mathfrak{Dxt}(A,B)$ such that

$$P_1\phi_1 \oplus \psi_2 = \pi(u)P_2\phi_2\pi(u^*) \oplus \psi_2.$$

Therefore,

$$(\phi_1, P_1) \oplus (\psi_1, 1) \text{ is homological to } (\phi_2, P_2) \oplus (\psi_2, 1)$$

hence $\mathrm{E}^1(A,B) \to \mathrm{Ext}(A,B)$, $[\phi, P] \mapsto [P\phi]$ is injective.

Now we prove that it is surjective. We need the following lemmas:

Lemma 8.26 ([4, Corollary 3.11])**.** *Let A and B be unital C^*-algebras, $J \subseteq B$ be an ideal in B. If either A, B or B/J is nuclear, then any unital completely positive map $\phi\colon A \to B/J$ has a unital completely positive lift $\hat{\phi}\colon A \to B$.*

Lemma 8.27 (Stinespring's dilation theorem)**.** *Let A be a separable unital C^*-algebra. Let B be a σ-unital C^*-algebra. Let $\phi\colon A \to \mathcal{M}(\mathbb{K} \otimes B)$ be a unital completely positive map. Then there exists a *-homomorphism $\rho\colon A \to \mathbb{M}_2\mathcal{M}(\mathbb{K} \otimes B)$ such that*

$$\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho(a) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now let $\phi\colon A \to \mathcal{Q}(\mathbb{K} \otimes B)$ be the Busby invariant of some extension. Since A is separable and nuclear, its unitisation $\tilde{A}$ is also separable nuclear and ϕ extends to a unital *-homomorphism $\tilde{\phi}\colon \tilde{A} \to \mathcal{Q}(\mathbb{K} \otimes B)$. By Lemma 8.26, there is a unital completely positive lift $\hat{\phi}\colon \tilde{A} \to \mathcal{M}(\mathbb{K} \otimes B)$. By Lemma 8.27, there exists a *-homomorphism $\rho\colon \tilde{A} \to \mathbb{M}_2\mathcal{M}(\mathbb{K} \otimes B)$ such that

$$\hat{\rho}(a) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho(a) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then one shows that $(\rho \circ i, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$ is an $(\tilde{A}, B)$-pair, where $i\colon A \hookrightarrow \tilde{A}$. And the image of $(\rho \circ i, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$ is ϕ. This finishes the proof. □

8.3 The isomorphism between KK_1 and Ext

Let $\mathbb{C}\ell_1$ be the first complex Clifford algebra. It is generated by a single self-adjoint unitary ϵ. There is a standard grading on $\mathbb{C}\ell_1$ by enforcing ϵ to be odd.

Definition 8.28. Let A and B be (graded) C*-algebras. Then

$$\mathrm{KK}_1(A,B) := \mathrm{KK}(A, B \hat{\otimes} \mathbb{C}\ell_1).$$

We have the following standard Hilbert C*-modules:

- Let B be a graded C*-algebra. Define $\mathcal{H}_{B\hat{\otimes}\mathbb{C}\ell_1} := \mathcal{H}_B \hat{\otimes} \mathbb{C}\ell_1$.
- Let B be an ungraded C*-algebra. Define $\hat{\mathcal{H}}_{B\otimes\mathbb{C}\ell_1} := \mathcal{H}_{B\otimes\mathbb{C}\ell_1} \oplus \mathcal{H}^{\mathrm{op}}_{B\otimes\mathbb{C}\ell_1}$ equipped with the obvious grading.
- Let B be a graded C*-algebra. Define $\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1} := \hat{\mathcal{H}}_B \hat{\otimes} \mathbb{C}\ell_1$.

And one can show that

$$\mathbb{B}(\hat{\mathcal{H}}_B) \cong (\mathbb{M}_2\mathcal{M}(\mathbb{K}\otimes B), \text{diagonal–off-diagonal grading}).$$

Lemma 8.29. *There is a graded isomorphism*

$$(\mathbb{M}_2\mathcal{M}(\mathbb{K}\otimes B), \textit{diagonal–off-diagonal grading}) \hat{\otimes} \mathbb{C}\ell_1 \cong (\mathbb{M}_2\mathcal{M}(\mathbb{K}\otimes B), \textit{trivial grading}) \otimes \mathbb{C}\ell_1.$$

As a corollary,

$$\mathbb{B}(\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}) \cong (\mathbb{M}_2\mathcal{M}(\mathbb{K}\otimes B), \textit{trivial grading}) \otimes \mathbb{C}\ell_1.$$

If A is separable and B is σ-unital. Then every class in $\mathrm{KK}(A, B \hat{\otimes} \mathbb{C}\ell_1)$ can be written as

$$(\hat{\mathcal{H}}_{B\otimes\mathbb{C}\ell_1}, \phi \hat{\otimes} 1, F \hat{\otimes} \epsilon),$$

for $\phi\colon A \to \mathbb{M}_2\mathcal{M}(\mathbb{K}\otimes B)$ and $F \in \mathbb{M}_2\mathcal{M}(\mathbb{K}\otimes B)$.

Lemma 8.30. *There is a bijection between* $\mathbb{E}(A, B \hat{\otimes} \mathbb{C}\ell_1)$ *and* $\mathcal{E}^1(A,B)$ *via*

$$(\mathcal{H}_{B\hat{\otimes}\mathbb{C}\ell_1}, \phi \hat{\otimes} 1, F \hat{\otimes} \epsilon) \longleftrightarrow (\phi, \frac{F+1}{2}).$$

Proof. Straightforward. □

Theorem 8.31. *The bijection in Lemma 8.30 induces an isomorphism of groups*

$$\mathrm{KK}_1(A,B) \xrightarrow{\cong} \mathrm{E}^1(A,B).$$

Proof. Actually, we have

$$\begin{array}{rcl} \text{Degenerate cycles} & \longleftrightarrow & \text{Degenerate pairs} \\ \text{Unitary equivalent cycles} & \longleftrightarrow & \text{Unitary equivalent pairs} \\ \text{Operator homotopic cycles} & \longleftrightarrow & \text{Homological pairs.} \end{array}$$

We write down the operator homotopy explicitly. Let (ϕ, P) and (ψ, Q) be homological pairs. Then after adding the degenerate cycles

$$(\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \phi \hat{\otimes} 1, -1 \hat{\otimes} \epsilon) \quad \text{and} \quad (\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \psi \hat{\otimes} 1, -1 \hat{\otimes} \epsilon),$$

we claim that the following two Kasparov modules:

$$(\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \phi \hat{\otimes} 1, -1 \hat{\otimes} \epsilon) \oplus (\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \phi \hat{\otimes} 1, (2P-1) \hat{\otimes} \epsilon)$$

and

$$(\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \psi \mathbin{\hat{\otimes}} 1, -1 \mathbin{\hat{\otimes}} \epsilon) \oplus (\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \psi \mathbin{\hat{\otimes}} 1, (2Q-1) \mathbin{\hat{\otimes}} \epsilon)$$

are operator homotopic. The operator homotopy is given by

$$\left(\hat{\mathcal{H}}_{B\hat{\otimes}\mathbb{C}\ell_1}, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix} 2P-1 & 2tPQ \\ 2tQP & 2t^2Q \end{pmatrix}\right).$$

Then the bijection $(\mathcal{H}_{B\hat{\otimes}\mathbb{C}\ell_1}, \phi \mathbin{\hat{\otimes}} 1, F \mathbin{\hat{\otimes}} \epsilon) \longleftrightarrow (\phi, \frac{F+1}{2})$ induces an (additive) isomorphism $\mathrm{KK}_1(A,B) \cong \mathrm{E}^1(A,B)$. Since $\mathrm{E}^1(A,B) \cong \mathrm{Ext}(A,B)$ as semigroups and $\mathrm{KK}_1(A,B)$ is a group, we conclude that $\mathrm{Ext}(A,B)$ is a group. □

May 10 and May 17, 2022

Categorical aspects of KK-theory

Speaker: Yuezhao Li (Leiden University)

Thanks to the Kasparov product, we may define the following Kasparov category KK:

Definition 9.1. The Kasparov category KK has the following data:

- Objects are separable C^*-algebras.
- An arrow $A \to B$ is an element in $\mathrm{KK}(A,B)$. The composition of two arrows is given by the Kasparov product.

It turns out that the Kasparov category KK has many better properties than the category of separable C^*-algebras $C^*\mathrm{Sep}$: KK is additive. Higson and Cuntz [12] noticed that this category can be characterised by its universal property. Meyer and Nest [19–21] observed that KK is triangulated and illutrated that many constructions and results (e.g. the Baum–Connes conjecture, the universal coefficient theorem) in KK-theory can be formally translated to the categorical language.

9.1 KK-theory as a universal functor

This section mainly follows Higson's article [12]. Our goal is to characterise KK-theory using universal properties. This is due to Higson and Cuntz.

Theorem 9.2 (Higson)**.** KK-*theory is the universal split-exact, homotopy-invariant and* $\mathbb{K}$-*stable functor.*

Remark 9.3. It is also possible (and sometimes desirable) to omit "homotopy-invariant" due to a result of Higson: $\mathbb{K}$-stable together with split-exact implies homotopy-invariant.

What is a universal functor? The following definition is given in [18].

Definition 9.4. Let (P) be a property defined on a category C. (We will always assume that C is a full subcategory of $C^*\mathrm{Alg}$). A *universal functor* subject to (P) consists of the following data:

- A category $\mathrm{Univ}_\mathrm{P}(\mathrm{C})$.
- A functor $U_\mathrm{P}\colon \mathrm{C} \to \mathrm{Univ}_\mathrm{P}(\mathrm{C})$.

such that:

- For any functor $\bar{F}\colon \mathrm{Univ}_\mathrm{P}(\mathrm{C}) \to \mathrm{D}$, the functor $F := \bar{F} \circ U_\mathrm{P}\colon \mathrm{C} \to \mathrm{D}$ satisfies (P).
- For any functor $F\colon \mathrm{C} \to \mathrm{D}$ satisfying (P), it factors as $F = \bar{F} \circ U_\mathrm{P}$ for a unique functor $\bar{F}\colon \mathrm{Univ}_\mathrm{P}(\mathrm{C}) \to \mathrm{D}$.

Example 9.5. Here are some examples of universal functors on the category $\mathrm{C} = C^*\mathrm{Alg}$.

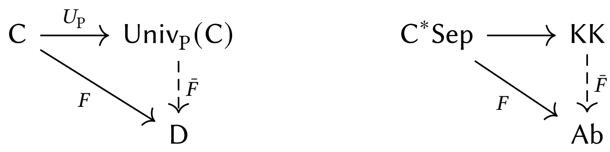

Figure 9.1: Universal functor $\mathsf{C} \xrightarrow{U_P} \mathrm{Univ}_P(\mathsf{C})$, $C^*\mathsf{Sep} \to \mathsf{KK}$.

- Let (P)="homotopy-invariant". Then $U_P\colon C^*\mathsf{Alg} \to \mathsf{HoC}^*\mathsf{Alg}$. An arrow from A to B in the category $\mathsf{HoC}^*\mathsf{Alg}$ is a homotopy class of *-homomorphisms $A \to B$.
- Let (P)="$\mathbb{K}$-stable". Then $U_P\colon C^*\mathsf{Alg} \to \mathsf{Corr}$. An arrow $A \rightsquigarrow B$ in the category Corr are *isomorphism classes* of C^*-correspondences from A to $B \otimes \mathbb{K}$. Recall that a C^*-correspondence from A to $B \otimes \mathbb{K}$ is a Hilbert $B \otimes \mathbb{K}$-module E together with a *non-degenerated* *-homomorphism $A \to \mathbb{B}_{B\otimes\mathbb{K}}(E)$.
- Let (P)="exact, homotopy-invariant and $\mathbb{K}$-stable". A functor $C^*\mathsf{Alg} \to \mathsf{Ab}$ is exact if it creates long exact sequences from extensions (without the requirement of being semi-split). A universal functor subject to (P) exists by some abstract and formal construction: notice that such a functor $C^*\mathsf{Alg} \to \mathsf{E}$ is also split-exact, so it factors uniquely through KK. Then one may realise this category by a quotient (localisation) of KK. The universal category E is the E-*theory* category, introduced by Connes and Higson [6].

There is a canonical functor $C^*\mathsf{Sep} \to \mathsf{KK}$. In a previous talk we have seen that KK-theory is split-exact, homotopy-invariant and $\mathbb{K}$-stable. What remains is to show that it is universal among all such functors.

Theorem 9.6. *Let $F\colon C^*\mathsf{Sep} \to \mathsf{Ab}$ be a split-exact, homotopy-invariant and $\mathbb{K}$-stable functor. Then there is a well-defined group homomorphism*

$$\mathrm{KK}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

for any separable C^*-algebras A and B. In particular: this construction is functorial in both A and B.

Proof. It is convenient to work in Cuntz's picture. Recall that in Cuntz's picture, a $\mathrm{KK_h}$-cycle (quasihomomorphism) from A to B is a pair (ϕ_+, ϕ_-) where $\phi_\pm\colon A \to \mathcal{M}(\mathbb{K} \otimes B)$ are *-homomorphisms, and such that $\phi_+(a) - \phi_-(a) \in \mathbb{K} \otimes B$ for all $a \in A$. Denote the class of quasihomomorphisms from A to B by $\mathbb{F}(A, B)$.

Given $\Phi := (\phi_+, \phi_-) \in \mathbb{F}(A, B)$. You might want to define the induced map in $\mathrm{Hom}(F(A), F(B))$ to be $F(\phi_+ - \phi_-)$ and use the fact that F is $\mathbb{K}$-stable. This does not work: $\phi_+ - \phi_-$ need not be a *-homomorphism, hence does not always induce a map between $F(A)$ and $F(\mathbb{K} \otimes B)$, so $F(\phi_+)(a) - F(\phi_-)(a)$ need not lie in $F(\mathbb{K} \otimes B)$.

We will have to use split-exactness of F. Define

$$A_\Phi := \{(a, x) \in A \oplus \mathcal{M}(\mathbb{K} \otimes B) \mid \phi_+(a) - x \in \mathbb{K} \otimes B\}.$$

There is an obvious extension of C^*-algebras

$$\mathbb{K} \otimes B \rightarrowtail A_\Phi \underset{\hat{\phi}_-}{\overset{\hat{\phi}_+}{\twoheadrightarrow}} A,$$

which splits by $\hat{\phi}_\pm := (\mathrm{id}, \phi_\pm)$. Another illuminating way is to consider the map $A \xrightarrow{\phi_+} \mathcal{M}(\mathbb{K} \otimes B) \twoheadrightarrow \mathcal{Q}(\mathbb{K} \otimes B)$ as the Busby invariant of some split extension.

Since $\hat{\phi}_\pm$ are *-homomorphisms. They define maps $F(\hat{\phi}_\pm)\colon F(A) \rightrightarrows F(A_\Phi)$. But F is split-exact, meaning that there is an isomorphism

$$F(A_\Phi) \cong F(A) \oplus F(\mathbb{K} \otimes B).$$

Define π to be the projection $F(A_\Phi) \to F(\mathbb{K} \otimes B)$. Then the map $\Phi_* \colon F(A) \to F(B)$ induced by $\Phi = (\phi_+, \phi_-)$ is given by

$$F(A) \xrightarrow{F(\hat{\phi}_+) - F(\hat{\phi}_-)} F(A_\Phi) \xrightarrow{\pi} F(\mathbb{K} \otimes B) \xrightarrow{\cong} F(B).$$

The last isomorphism is due to $\mathbb{K}$-stability. Finally, by homotopy-invariance of F one can show that the map is well-defined on the level of $\mathrm{KK_h}$-groups. □

We define Φ_* to be the map in $\mathrm{Hom}(F(A), F(B))$ induced by $[\Phi] \in \mathrm{KK}(A, B)$.

Corollary 9.7. *There is a well-defined functor* $\bar{F} \colon \mathsf{KK} \to \mathsf{Ab}$ *in Figure 9.1:*

- *Object level:* $\bar{F}(A) \coloneqq F(A)$.
- *Arrow level:* $\bar{F}([\Phi]) \coloneqq \Phi_*$.

It remains to prove the uniqueness.

Proposition 9.8. *Let* $x \in F(A)$*. Then there exists a unique natural transformation*

$$\alpha \colon \mathrm{KK}(A, -) \Rightarrow F$$

such that $\alpha_A(1_A) = x$.

Proof. There is a unique natural transformation satisfying all the required properties:

$$\begin{array}{ccc} \mathrm{KK}(A,A) & \xrightarrow{[\Phi]} & \mathrm{KK}(A,B) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ F(A) & \xrightarrow[\Phi_*]{} & F(B) \end{array} \qquad \begin{array}{ccc} 1_A & \xmapsto{[\Phi]} & [\Phi] \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ x & \xmapsto[\Phi_*]{} & \Phi_*(x) \end{array}$$

so $\alpha_B([\Phi]) \coloneqq \Phi_*(x)$. □

Corollary 9.9. *The Kasparov product is unique. That is, there is a unique bilinar map*

$$\mathrm{KK}(A, B) \otimes \mathrm{KK}(B, C) \xrightarrow{\otimes_B} \mathrm{KK}(A, B)$$

which is functorial in A, B and C and satisfies $1_B \otimes_B x = x$, $y \otimes_B 1_B = y$ *for all suitable x, y and B.*

Proof. Suppose there is another Kasparov product, denoted by $\otimes'$. Consider the natural transformations

$$\begin{aligned} \alpha &\colon \mathrm{KK}(C, -) \Rightarrow \mathrm{KK}(A, -), \quad z \mapsto (x \otimes' y) \otimes z, \\ \beta &\colon \mathrm{KK}(C, -) \Rightarrow \mathrm{KK}(A, -), \quad z \mapsto x \otimes' (y \otimes z) \end{aligned}$$

for some given fixed $x \in \mathrm{KK}(A, B)$ and $y \in \mathrm{KK}(B, C)$. For $1_C \in \mathrm{KK}(C, C)$:

$$(x \otimes' y) \otimes 1 = x \otimes' y = x \otimes' (y \otimes 1).$$

So $\alpha_C(1_C) = \beta_C(1_C) = x \otimes' y$. By previous lemma, this forces $\alpha = \beta$. Therefore,

$$(x \otimes' y) \otimes z = x \otimes' (y \otimes z)$$

for all $x \in \mathrm{KK}(A, B)$, $y \in \mathrm{KK}(B, C)$ and $z \in \mathrm{KK}(C, D)$. Then

$$x \otimes' y = (x \otimes 1_B) \otimes' (1_B \otimes y) = x \otimes (1_B \otimes' 1_B) \otimes y = x \otimes 1_B \otimes y = x \otimes y.$$ □

Proof of Theorem 9.2. Define $\bar{F}$ as in Corollary 9.7. This is a functor. We claim it is unique. Notice that there is a natural transformation

$$\mathrm{KK}(A, -) \Rightarrow \mathrm{Hom}(F(A), F(-)),$$

which by functoriality, sends 1_A to $1_{F(A)}$. This uniquely charaterises the natural transformation, hence the functor $\bar{F}$, by the previous lemma. □

Remark 9.10. Since K-theory is also split-exact, homotopy-invariant and stable, we have well-defined maps

$$\mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$$

and

$$\mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B))$$

which are given by the Kasparov product. What do we know about these maps in general? Are they surjective or an isomorphism? This shall be answered by the universal coefficient theorem.

9.2 KK-theory as a triangulated category

9.2.1 Universal coefficient theorem

The universal coefficient theorem (UCT) allows us to understand the map $\mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$ and $\mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B))$. Let $\mathrm{K}_*(A) := \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$ be the $\mathbb{Z}/2$-graded abelian group with the obvious grading.

Theorem 9.11. *Let A and B be separable C^*-algebras and A lies in the bootstrap class (Definition 9.15). Then there is a short exact sequence of abelian groups*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(A)) \to \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0.$$

which splits, but not naturally.

The functor $\mathrm{Ext}_{\mathbb{Z}}$ is the (first) right derived functor of Hom in the category of abelian groups (=$\mathbb{Z}$-modules). We recall the definition.

Definition 9.12. Let R be a commutative ring.

- Let A be an R-module. A projective resolution of A is an exact chain complex of R-modules

  $$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0,$$

  such that every P_i is projective.
- Let A and B be R-modules. Take a projective resolution

  $$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

  of A. Then $\mathrm{Ext}^n_R(A, B)$ is defined as the cohomology of the following chain complex

  $$\cdots \leftarrow \mathrm{Hom}_R(P_2, B) \leftarrow \mathrm{Hom}_R(P_1, B) \leftarrow \mathrm{Hom}_R(P_0, B) \leftarrow 0,$$

  $\mathrm{Hom}_R(P_i, B)$ is the abelian group of R-modules maps $P_i \to B$.

Remark 9.13.

1. $\mathrm{Ext}^n_R(-, B)\colon \mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Ab}$ is a functor. It is the n-th right derived functor of the functor $\mathrm{Hom}_R(-, B)\colon \mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Ab}$.
2. $\mathrm{Ext}^0_R(-, B) = \mathrm{Hom}_R(-, B)$.
3. $\mathrm{Ext}^n_R(A, B)$ does not depend on the choice of the projective resolution of A.
4. If A is projective, then $\mathrm{Ext}^n_R(A, B) = 0$ for $n \geq 2$. This is because A has a length-one projective resolution

   $$0 \to A \to A \to 0.$$

   Then by definition, $\mathrm{Ext}^n_R(A, B) = 0$ for all $n \geq 2$. In this case we also write $\mathrm{Ext}_R := \mathrm{Ext}^1_R$.

Lemma 9.14. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Let D be another R-module. Then there is a natural long exact sequence*

$$\begin{aligned}\cdots \leftarrow \mathrm{Ext}^n_R(A,D) \leftarrow \mathrm{Ext}^n_R(B,D) \leftarrow \mathrm{Ext}^n_R(C,D) \leftarrow \cdots \\ \leftarrow \mathrm{Ext}^1_R(C,D) \leftarrow \mathrm{Hom}_R(A,D) \leftarrow \mathrm{Hom}_R(B,D) \leftarrow \mathrm{Hom}_R(C,D) \leftarrow 0.\end{aligned}$$

Definition 9.15. The bootstrap class N is the smallest class of *nuclear, separable* C^*-algebras, satisfying:

(N1) $\mathbb{C} \in N$.

(N2) N is closed under countable direct limit.

(N3) N is closed under extension.

(N4) N is closed under KK-equivalence.

Remark 9.16. Many well-known C^*-algebras belong to N:

$$\begin{aligned}\mathbb{C} \in N &\overset{\text{(N3)}}{\Longrightarrow} \mathrm{C}_0(\mathbb{R}^n), \mathrm{C}([0,1]^n) \in N \overset{\text{(N3)}}{\Longrightarrow} \mathrm{C}(X) \in N, \text{ for any finite simplicial complex } X \\ &\overset{\text{(N2)}}{\Longrightarrow} \mathrm{C}(X) \in N, \text{ for any compact } X = \text{unital commutative } \mathrm{C}^*\text{-algebras} \\ &\overset{\text{(N3)}}{\Longrightarrow} \mathrm{C}_0(X) \in N, \text{for any locally compact } X = \text{commutative } \mathrm{C}^*\text{-algebras.}\end{aligned}$$

Where does the term $\mathrm{Ext}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B))$ come from? The UCT for cohomology gives us some motivation. Recall that

Theorem 9.17 (UCT in cohomology)**.** *Let (C_*, d_*) be a chain complex of free abelian groups. Let G be any abelian group. Then there is a short exact sequence of abelian groups:*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(\mathrm{H}_*(C_*), G) \to \mathrm{H}^*(C_*, G) \to \mathrm{Hom}(H_*(C_*), G) \to 0.$$

Sketch of the proof. Let

$$\begin{aligned}B_* &:= \operatorname{im} d_{*+1} \text{ be the chain of boundaries with the zero differential.} \\ Z_* &:= \ker d_* \text{ be the chain of cycles with the zero differential.}\end{aligned}$$

Then there are short exact sequences

$$0 \to Z_* \to C_* \xrightarrow{d_*} B_{*-1} \to 0$$
$$0 \to B_* \xrightarrow{i} Z_* \to \mathrm{H}_*(C_*) \to 0$$

Notice that since C_* is a chain complex of free abelian groups, then both Z_* and B_* are chain complex of abelian group. (But $\mathrm{H}_*(C_*)$ may not be!) The first short exact sequence

$$0 \to Z_* \to C_* \xrightarrow{d_*} B_{*-1} \to 0,$$

together with Lemma 9.14 and the fact that B_{*-1} is free (=projective in the case of abelian groups) yields

$$0 \leftarrow \mathrm{Hom}(Z_*, G) \leftarrow \mathrm{Hom}(C_*, G) \leftarrow \mathrm{Hom}(B_{*-1}, G) \leftarrow 0.$$

This new short exact sequence induces a long exact sequence in cohomology. Since the complexes $\mathrm{Hom}(Z_*, G)$ and $\mathrm{Hom}(B_*, G)$ both have zero differential (!), we identify $\mathrm{H}^*(\mathrm{Hom}(Z_*, G))$ (or $\mathrm{H}^*(\mathrm{Hom}(B_*, G))$) with $\mathrm{Hom}(Z_*, G)$ (or $\mathrm{Hom}(B_*, G)$) itself. The long exact sequence reads

$$\cdots \leftarrow \mathrm{Hom}(B_*, G) \xleftarrow{i^*} \mathrm{Hom}(Z_*, G) \leftarrow \mathrm{H}^*(C_*, G) \leftarrow \mathrm{Hom}(B_{*-1}, G) \xleftarrow{i^*} \mathrm{Hom}(Z_{*-1}, G) \leftarrow \cdots$$

and is cut down to the following short exact sequence

$$0 \to \operatorname{coker} i^* \to \mathrm{H}^*(C_*, G) \to \ker i^* \to 0.$$

It suffices to identify $\operatorname{coker} i^*$ and $\ker i^*$ with the corresponding terms in UCT. Consider the second short exact sequence

$$0 \to B_* \xrightarrow{i} Z_* \to \mathrm{H}_*(C_*) \to 0$$

which by Lemma 9.14 gives a long exact sequence

$$0 \leftarrow \mathrm{Ext}_{\mathbb{Z}}(\mathrm{H}_*(C_*), G) \leftarrow \mathrm{Hom}(B_*, G) \xleftarrow{i^*} \mathrm{Hom}(Z_*, G) \leftarrow \mathrm{Hom}(\mathrm{H}_*(C_*), G) \leftarrow 0.$$

So $\mathrm{Ext}_{\mathbb{Z}}(\mathrm{H}_*(C_*), G) \cong \operatorname{coker} i^*$ and $\mathrm{Hom}(\mathrm{H}_*(C_*), G) \cong \ker i^*$. □

Remark 9.18. The Ext^1 appears precisely because $\mathrm{H}^n(C_*, G)$ is an abelian group for every n, and any abelian group has a projective resolution of length one, so that we may identify Ext with a suitable cokernel. This motivates us to study the projective resolution of $\mathrm{K}_*(A)$. Unlike UCT in cohomology, in which we do not leave the category Ab (or more precisely: the category Kom(Ab) of chain complexes in Ab). For UCT in KK, we need to work with the Kasparov category KK, which is only additive but not abelian; and we need to find a suitable way to lift a projective resolution of $\mathrm{K}_*(A)$ to a "resolution" of A. All these can be made precise by studying the triangulated structure of KK.

9.2.2 Triangulated categories

Definition 9.19.
- Let T be a (locally small) category. It is *additive*, if the followings hold:
  1. T has zero object.
  2. T has finite biproduct (i.e. finite products coincide with finite coproducts)
  3. $\mathsf{T}(A, B)$ is an abelian group for all A, B. ($\mathsf{T}(A, B)$ is the set of arrows from A to B.)
  4. The composition of arrows is a group homomorphism.
- Let T be an additive category. A *suspension functor* is an additive automorphism $\Sigma\colon \mathsf{T} \to \mathsf{T}$. (Additive means $\Sigma(f+g) = \Sigma(f) + \Sigma(g)$). A *stable additive category* is an additive category T together with a suspension functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$.
- A *triangle* in a stable additive category (T, Σ) is a diagram

  $$\Sigma C \to A \to B \to C$$

  in this category. A *morphism* between triangles $\Sigma C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and $\Sigma C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ is a diagram

  $$\begin{array}{ccccccc} \Sigma C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ \downarrow{\scriptstyle \Sigma\gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \Sigma C' & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & B' & \xrightarrow[h']{} & C'. \end{array}$$

  Two triangles are *isomorphic* if there exists a pair of invertible morphisms between them.

Definition 9.20. A *triangulated structure*[3] on an additive category T consists of a suspension functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$, together with a collection of triangles called *exact triangles*, satisfying the following axioms:

[3] Actually, what I am writing down is a cotriangulated structure, which is the structure on the opposite category of a triangulated category. As mentioned in [21], the triangulated structure actually lies in $\mathrm{KK}^{\mathrm{op}}$. But this makes quite little difference: the opposite category of a triangulated category is also triangulated. This makes only some notational trouble.

(TR0) Any triangle isomorphic to an exact triangle is exact. The triangle

$$\Sigma A \to 0 \to A \xrightarrow{\mathrm{id}} A$$

is exact.

(TR1) Given any arrow $f\colon B \to C$, there exists an exact triangle

$$\Sigma C \to A \to B \xrightarrow{f} C.$$

(TR2) "Rotation axiom". $\Sigma C \to A \to B \to C$ is exact iff $\Sigma B \to \Sigma C \to A \to B$ is exact.

(TR3) Given two exact triangles $\Sigma C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and $\Sigma C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$. If there exist an arrows $\beta\colon B \to B'$ and $\gamma\colon C \to C'$ such that the diagram

$$\begin{array}{ccccccc} \Sigma C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ \downarrow{\scriptstyle \Sigma\gamma} & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \Sigma C' & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & B' & \xrightarrow[h']{} & C' \end{array}$$

commutes. Then there exists (maybe not uniquely!) an arrow $\alpha\colon A \to A'$ such that the diagram becomes a morphism of triangles, that is, the following diagram commutes:

$$\begin{array}{ccccccc} \Sigma C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ \downarrow{\scriptstyle \Sigma\gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \Sigma C' & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & B' & \xrightarrow[h']{} & C'. \end{array}$$

(TR4) "Octahedron axiom". See [20, Appendix A].

Example 9.21. Let C be an abelian category (e.g. Ab). Define the category of chain complexes in C

$$\mathsf{Kom}(\mathsf{C}) := (\text{Chain complexes in } \mathsf{C}, \text{chain maps})$$

and its homotopy category

$$\mathsf{HoKom}(\mathsf{C}) := (\text{Homotopy classes of chain complexes in } \mathsf{C}, \text{homotopy classes of chain maps}).$$

Then $\mathsf{HoKom}(\mathsf{C})$ is triangulated. The suspension is the shift functor $A_* \mapsto A[1]_*$ where $A[1]_n := A_{n+1}$. The exact triangles are mapping cone triangles.

Definition 9.22. Let T be a triangulated category. An additive functor $F\colon \mathsf{T} \to \mathsf{Ab}$ is called a *homological functor*, if it creates long exact sequences from exact triangles. More precisely, given an exact triangle $\Sigma C \to A \to B \to C$ there is an induced long exact sequence

$$\cdots \to F(\Sigma B) \to F(\Sigma C) \to F(A) \to F(B) \to F(C) \to F(\Sigma^{-1} A) \to \cdots.$$

A cohomological functor is defined likewise by replacing T with T^{op} and reversing the arrows.

Lemma 9.23. *For any object D in T, the functor $\mathsf{T}(D,-)\colon \mathsf{T} \to \mathsf{Ab}$ is homological.*

Proof. Let $\Sigma C \to A \xrightarrow{f} B \xrightarrow{g} C$ be an exact triangle. It suffices to check the exactness of

$$\mathsf{T}(D,A) \xrightarrow{f_*} \mathsf{T}(D,B) \xrightarrow{g_*} \mathsf{T}(D,C).$$

Exactness at all other places follows from the rotation axiom (TR2).

- $\operatorname{im} f_* \subseteq \ker g_*$. By functoriality of $\mathsf{T}(D, -)$, it suffices to prove that $g \circ f = 0$. Consider the diagram

$$\begin{array}{ccccccc} \Sigma C & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle \Sigma\,\mathrm{id}}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{id}} \\ \Sigma C & \longrightarrow & 0 & \longrightarrow & C & \xrightarrow[\mathrm{id}]{} & C \end{array}$$

with all the solid-line arrows. By (TR3), the diagram has to be a morphism of triangles, so we obtain a map $A \to 0$, which has to be the zero arrow. Then the diagram commutes implies that $g \circ f = \mathrm{id} \circ g \circ f = 0$.

- $\operatorname{im} f_* \supseteq \ker g_*$. By (TR0) and the rotation axiom (TR2), the diagram

$$0 \to D \xrightarrow{\mathrm{id}} D \to 0$$

is also an exact triangle. Now suppose $\phi \in \ker g_*$. That is, $\phi \colon D \to B$ satisfies $g \circ \phi = 0$. Then the following diagram (with all solid-line arrows) commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & D & \xrightarrow{\mathrm{id}} & D & \longrightarrow & 0 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} \\ \Sigma C & \longrightarrow & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C. \end{array}$$

Now by (TR3), there exists an dash-line arrow $\tilde{\phi} \colon D \to A$ makeing the diagram a morphism of triangles. Then we obtain $\phi = \phi \circ \mathrm{id} = f \circ \tilde{\phi}$ since the diagram commutes. Therefore $\phi \in \operatorname{im} f_*$. □

Theorem 9.24. KK *is triangulated. Its suspension is* $A \mapsto \mathrm{S}A$ *and its exact triangles are triangles which are isomorphic to mapping cone triangles* $\mathrm{S}A \to \mathrm{C}_f \to A \xrightarrow{f} B$ *for a suitable* **-homomorphism* $f \colon A \to B$.

Remark 9.25. Equivalently, the exact triangles can also be defined in terms of *extension triangles*

$$\mathrm{S}Q \to I \to E \to Q$$

where $I \overset{i}{\rightarrowtail} E \overset{q}{\twoheadrightarrow} Q$ is a semi-split extension of C*-algebras. The extension defines a class in $\mathrm{KK}_1(Q, I) \cong \mathrm{KK}(\mathrm{S}Q, I)$ and this is the arrow $\mathrm{S}Q \to I$. Mapping cone extensions are semi-split, hence they are all extension triangles. Conversely, given an extension triangle, there is an KK-equivalence $I \sim_{\mathrm{KK}} \mathrm{C}_q$. So any extension triangle is isomorphic to a mapping cone triangle.

Proof. We need to check the axioms (TR0)–(TR4). The proof can be found in [20, Appendix A]. Here we sketch the proof of (TR1)–(TR3)[4].

For (TR0), notice that the mapping cone $\mathrm{C}_{\mathrm{id}} = \mathrm{C}A$ is the cone of A. This is a contractible C*-algebra and hence KK-equivalent to 0. Then we have an isomorphism between $\mathrm{S}A \to 0 \to A \to A$ and the mapping cone triangle of $\mathrm{id} \colon A \to A$.

For (TR1). If $f \in \mathrm{KK}(B, C)$ is induced by a *-homomorphism, then we simply use its mapping cone triangle. In the general case, we need to identify f with a *-homomorphism up to KK-equivalence. This can be done using Cuntz's picture [9]: any element in $\mathrm{KK}(B, C)$ can be represented by a *-homomorphism $\mathrm{q}B \to C \otimes \mathbb{K}$, where $\mathrm{q}B$ is a C*-algebra that is KK-equivalent to B. Then we have several KK-equivalences $\mathrm{q}B \sim_{\mathrm{KK}} B$ and $C \otimes \mathbb{K} \sim_{\mathrm{KK}} C$. Together with the axiom (TR3), they provide an isomorphism between two triangles, one of which is a mapping cone triangle and hence exact.

For (TR2). It suffices to look at mapping cone triangles by (TR0) and (TR1). Given a mapping cone triangle $\mathrm{S}B \to \mathrm{C}_f \xrightarrow{\pi_A} A \xrightarrow{f} B$, the triangle $\mathrm{S}A \to \mathrm{S}B \to \mathrm{C}_f \xrightarrow{\pi_A} A$ is isomorphic to a mapping cone triangle because $\mathrm{S}B$ is homotopy equivalent to C_{π_A}, see the proof of Theorem 6.15.

[4] Sketching is already quite long...

For (TR3), this is done by some suitable gluing. Given two mapping cone triangles together with arrows

$$\begin{array}{ccccccc} \mathrm{S}B & \longrightarrow & \mathrm{C}_f & \longrightarrow & A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \mathrm{S}\beta} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathrm{S}B' & \longrightarrow & \mathrm{C}_{f'} & \longrightarrow & A' & \xrightarrow[f']{} & B' \end{array}$$

where $f\colon A \to B$ and $f'\colon A' \to B'$ are *-homomorphisms, $\alpha \in \mathbb{E}(A, A')$ and $\beta \in \mathbb{E}(B, B')$ are Kasparov modules which represent KK-theory classes. The diagram commutes implies that $f^*(\beta)$ are $f'_*(\alpha)$ are homotopic Kasparov modules. Let $H \in \mathbb{E}(A, \mathrm{I}B)$ be a homotopy connecting them. Define

$$\mathrm{I}_{f'} := \{(a, \phi) \in A' \oplus \mathrm{I}B' \mid f'(a) = \mathrm{ev}_1(\phi)\}$$

to be the mapping cylinder of f'. Then $\alpha \oplus H \in \mathbb{E}(A, \mathrm{I}_{f'})$ because $f'_*(\alpha) = (\mathrm{ev}_1)_* H$.

There is a homotopy equivalence between the C*-algebras A' and $\mathrm{I}_{f'}$ given by

$$\iota\colon A' \xrightarrow{\cong} \mathrm{I}_{f'}, \quad a \mapsto (a, t \mapsto f'(a))$$
$$\pi\colon \mathrm{I}_{f'} \xrightarrow{\cong} A', \quad (a, \phi) \mapsto a.$$

And we may use this map $\pi\colon \mathrm{I}_{f'} \xrightarrow{\cong} A'$ to get $\pi_*(\alpha \oplus H) \in \mathbb{E}(A, A')$. We pullback it to $\pi_A^* \pi_*(\alpha \oplus H) \in \mathbb{E}(\mathrm{C}_f, A')$.

Now consider $\tau_{\mathrm{C}_0(0,1]}\beta \in \mathbb{E}(\mathrm{C}B, \mathrm{C}B')$. Use $\pi_{\mathrm{C}B}\colon \mathrm{C}_f \to \mathrm{C}B$ to pull it back to $\pi_{\mathrm{C}B}^*(\tau_{\mathrm{C}_0(0,1]}\beta) \in \mathbb{E}(\mathrm{C}_f, \mathrm{C}B')$. Then the Kasparov module

$$\pi_A^* \pi_*(\alpha \oplus H) \oplus \pi_{\mathrm{C}B}^*(\tau_{\mathrm{C}_0(0,1]}\beta) \in \mathbb{E}(\mathrm{C}_f, \mathrm{C}_{f'})$$

defines the desired in $\mathrm{KK}(\mathrm{C}_f, \mathrm{C}_{f'})$. □

9.2.3 Proof of the Universal Coefficient Theorem

Homological algebra in non-abelian categories is always relative, *we need additional structure to get started*[5]. In a triangulated category T this can be worked out by *ideals*. An ideal in a triangulated category T is a family of subgroups $\mathsf{I}(A, B) \subseteq \mathsf{T}(A, B)$ for all A, B, satisfying

$$\mathsf{T}(C, D) \circ \mathsf{I}(B, C) \circ \mathsf{T}(A, B) \subseteq \mathsf{I}(A, D)$$

for all A, B, C, D. In our situation, we only care about the ideal

$$\ker \mathrm{K}_* := \{f \mid \mathrm{K}_*(f) = 0\}.$$

Given any separable C*-algebra A, $\mathrm{K}_*(A)$ is a countably-generated $\mathbb{Z}/2$-graded abelian group. So $\mathrm{K}_*(A)$ has a length-one projective resolution. Given such a resolution, can we lift it to some kind of resolution of C*-algebras?

Definition 9.26.

- An exact triangle $\Sigma C \to A \to B \to C$ in KK is a K_**-exact triangle*, if

$$0 \to \mathrm{K}_*(A) \to \mathrm{K}_*(B) \to \mathrm{K}_*(C) \to 0$$

  is a short exact sequence.
- A homological functor $F\colon \mathsf{KK} \to \mathsf{Ab}$ is K_**-exact functor*, if it maps K_*-exact triangles to short exact sequences.
- A C*-algebra A is called K_*-projective if $\mathrm{KK}(A, -)$ is K_*-exact.

Example 9.27.

- The exact triangles defined by split extensions are K_*-exact.

[5] c.f. [21, Introduction]

- K_* is K_*-exact.
- $\mathbb{C}$ and $C_0(\mathbb{R})$ are K_*-projective because $\mathrm{KK}(\mathbb{C}, -) = \mathrm{K}_0$ and $\mathrm{KK}(C_0(\mathbb{R}), -) = \mathrm{K}_1$.

Lemma 9.28 ([21, Theorem 3.41]). *A is K_*-projective iff $\mathrm{K}_*(A)$ is projective and there is a natural isomorphism*

$$\mathrm{KK}(A, B) \cong \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$$

for all B.

Lemma 9.29. *Every separable C^*-algebra has a K_*-projective resolution of length-one.*

Idea of the proof. A countably-generated projective $\mathbb{Z}/2$-graded abelian group has the form

$$\bigoplus_{I_0} \mathbb{Z} \oplus \bigoplus_{I_1} \mathbb{Z}$$

with $\bigoplus_{I_0} \mathbb{Z}$ the even part and $\bigoplus_{I_1} \mathbb{Z}$ the odd part. Define

$$\mathrm{K}^\dagger\left(\bigoplus_{I_0} \mathbb{Z} \oplus \bigoplus_{I_1} \mathbb{Z}\right) := \bigoplus_{I_0} \mathbb{C} \oplus \bigoplus_{I_1} C_0(\mathbb{R}).$$

This is a functor which restricts to an equivalence between countably-generated projective $\mathbb{Z}/2$-graded abelian group and K_*-projective objects in KK ([21, Theorem 3.39, Theorem 3.41]). Now given a projective resolution

$$0 \to H_1 \to H_0 \to \mathrm{K}_*(A) \to 0$$

of $\mathrm{K}_*(A)$. Applying the functor $\mathrm{K}^\dagger$ we obtain a K_*-projective resolution

$$0 \to \mathrm{K}^\dagger(H_1) \to \mathrm{K}^\dagger(H_0) \to A \to 0.$$ □

Now we sketch the proof of UCT.

Proof of UCT.

- Given a C^*-algebra A, by Lemma 9.29 there is a length-one K_*-projective resolution

$$0 \to P_1 \xrightarrow{d} P_0 \to A \to 0.$$

- A Lemma:

  Lemma 9.30. *If A is in the bootstrap class, then the extension $P_1 \rightarrowtail P_0 \twoheadrightarrow A$ embeds in an exact triangle*

$$SA \to P_1 \to P_0 \to A.$$

- Apply the cohomological functor $\mathrm{KK}(-, B)$ (Lemma 9.23) to obtain a long exact sequence

$$\begin{array}{ccccc} \mathrm{KK}(P_1, B) & \xleftarrow{d^*} & \mathrm{KK}(P_0, B) & \longleftarrow & \mathrm{KK}(A, B) \\ \downarrow & & & & \uparrow \\ \mathrm{KK}_1(A, B) & \longrightarrow & \mathrm{KK}_1(P_0, B) & \xrightarrow[d^*]{} & \mathrm{KK}_1(P_1, B) \end{array}$$

  which cuts down to the short exact sequence

$$0 \to \operatorname{coker} d^* \to \mathrm{KK}(A, B) \to \ker d^* \to 0. \tag{5}$$

- By Lemma 9.28, identify the following d^*'s:

$$\begin{array}{ccc} \mathrm{KK}(P_0, B) & \xrightarrow{d^*} & \mathrm{KK}(P_1, B) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}(\mathrm{K}_*(P_0), \mathrm{K}_*(B)) & \xrightarrow[d^*]{} & \mathrm{Hom}(\mathrm{K}_*(P_1), \mathrm{K}_*(B)) \end{array}$$

- By Lemma 9.14 and the fact that $\mathrm{K}_*(P_0)$ is projective, we have a long exact sequence

$$0 \leftarrow \mathrm{Ext}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \leftarrow \mathrm{Hom}(\mathrm{K}_*(P_1), \mathrm{K}_*(B)) \xleftarrow{d^*} \mathrm{Hom}(\mathrm{K}_*(P_0), \mathrm{K}_*(B)) \leftarrow \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \leftarrow 0.$$

So we may identify $\mathrm{Ext}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \cong \operatorname{coker} d^*$ and $\mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \cong \ker d^*$. Substitute the corresponding terms in (5) we obtain the UCT. □

May 24, 2022

Finite summability in K-homology

Speaker: Dimitris Gerontogiannis (Leiden University)

10.1 Historical review of K-homology

We have had a glance at K-homology before (Section 6.2.3). There are several different approaches to it.

10.1.1 Abstract definition of K-homology

Definition 10.1. Let X be a finite CW-complex. Choose an embedding $X \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{S}^{n+1}$. The compliment $\mathbb{S}^{n+1} \setminus X$ deformation retracts to a dual complex $\mathrm{D}_n X$ up to suspension. That is, we have a homotopy equivalence $\Sigma(\mathbb{S}^{n+1} \setminus X) \simeq \Sigma(\mathrm{D}_n X)$. The space $\mathrm{D}_n X$ is called the *Spanier–Whitehead dual* of X. Define the K-homology of X to be

$$\mathrm{K}_0(X) := \mathrm{K}^0(\mathrm{D}_n X).$$

This definition does not depend on n, not even depend on the choice $X \hookrightarrow \mathbb{S}^{n+1}$.

10.1.2 Atiyah's approach (1970s)

Let M be a closed, smooth manifold. Let D be an elliptic operator on M. There exists a well-defined index map

$$\mathrm{Index}_D\colon\ \mathrm{K}^0(M) \longrightarrow \mathbb{Z}, \qquad [E] \longmapsto \mathrm{ind}(D_E),$$

where D_E is the operator D "twisted" by the vector bundle E. The construction uses a connection, but it turns out that $\mathrm{ind}(D_E)$ does not depend on the choice of the connection.

Similarly, if X is a finite CW-complex. Let $P \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded operator between two Hilbert spaces (say, the L^2-sections of some vector bundles) such that $[P, f] \in \mathbb{K}$ for all $f \in \mathrm{C}(X)$, and there exists a parametrix Q such that $1 - PQ \in \mathbb{K}$ and $1 - QP \in \mathbb{K}$. Then we say P is an elliptic operator on X, and there is a well-defined index map

$$\mathrm{Index}_P\colon\ \mathrm{K}^0(M) \longrightarrow \mathbb{Z}, \qquad [E] \longmapsto \mathrm{ind}(P_E).$$

Define $\mathrm{Ell}(X)$ to be the set of all *elliptic* operators on X. The construction above gives rise to a map

$$\mathrm{Ell}(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}^0(X), \mathbb{Z}).$$

There is also a map

$$\mathrm{K}_0(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}^0(X), \mathbb{Z}),$$

and by the universal coefficient theorem in (generalised) (co)homology theories: this is a rational isomorphism. So we have a map

$$\mathrm{Ell}(X) \longrightarrow \mathrm{K}_0(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

But Atiyah realised that this map actually factors through a map $\mathrm{Ell}(X) \longrightarrow \mathrm{K}_0(X)$.

10.1.3 Brown–Douglas–Fillmore theory (1960s)

There is a close relation between extensions of C^*-algebras and K-homology.

Theorem 10.2 (Weyl–von Neumann)**.** *Let $\mathcal{H}$ be a separable Hilbert space, $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ be self-adjoint operators. Then*

$$T_1 \text{ is essentially unitarily equivalent to } T_2 \quad \text{iff} \quad T_1 \text{ and } T_2 \text{ have the same essential spectrum.}$$

One tries to extend this result to (essentially) normal operators. But there can be obstructions for two such operators (with the same essential spectrum) to be essentially unitarily equivalent. One of them is the Fredholm indices of the Fredholm alternatives of T_i. Brown–Douglas–Fillmore (BDF) theory claims that this is the only obstruction.

Theorem 10.3 (Brown–Douglas–Fillmore)**.** *Let $\mathcal{H}$ be a separable Hilbert space, $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ be essentially normal operators with the same essential spectrum X. Then*

$$T_1 \text{ is essentially unitarily equivalent to } T_2 \quad \text{iff} \quad \operatorname{ind}(T_1 - \lambda) = \operatorname{ind}(T_2 - \lambda) \text{ for all } \lambda \in \mathbb{C} \setminus X.$$

Idea of the proof. Define $\operatorname{Ext}(X)$ to be the set

$$\left\{ \begin{array}{c} \text{Essentially unitary equivalence classes of essentially normal operators} \\ \text{with essential spectrum } X \end{array} \right\}.$$

Let $[T] \in \operatorname{Ext}(X)$ where T is essentially normal and has essential spectrum X. Then there is a well-defined group homomorphism

$$K^1(X) \to \mathbb{Z}, \qquad [z - \lambda] \mapsto \operatorname{ind}(T - \lambda).$$

By universal coefficient theorem, this induces an isomorphism $\operatorname{Ext}(X) \to \operatorname{Hom}_{\mathbb{Z}}(K^1(X), \mathbb{Z})$. □

10.1.4 Kasparov's approach (1970s)

Definition 10.4. Let A be a C^*-algebra. An odd Fredholm module over A is $(\mathcal{H}, \rho, F)$ where

- $\mathcal{H}$ is a separable Hilbert space.
- $\rho \colon A \to \mathbb{B}(\mathcal{H})$ is a *-homomorphism.
- $F \in \mathbb{B}(\mathcal{H})$ satisfies

$$[F, \rho(a)],\ \rho(a)(F^2 - 1),\ \rho(a)(F - F^*) \in \mathbb{K}(\mathcal{H}), \qquad \text{for all } a \in A.$$

An even Fredholm module over A is the $\mathbb{Z}/2$-graded version.

Remark 10.5. Using some similar trick that we have mentioned before, we can take $F = F^*$ and $F^2 = 1$. In this case we call the Fredholm module *normalised*.

Definition 10.6. The K-homology groups $K^0(A)$ (respectively, $K^1(A)$) is defined as the abelian group generated by even (respectively, odd) Fredholm modules up to operator homotopy (Section 4.3.2). This means that X and X' define the same K-homology class iff there is a degenerate X'' such that $X \oplus X''$ is operator homotopic to $X' \oplus X''$.

Definition 10.7. Let $(\mathcal{H}, \rho, F)$ be an even or odd Fredholm module. Define the index map $\operatorname{Index}_{(\mathcal{H},\rho,F)}$ as follows:

- If $(\mathcal{H}, \rho, F)$ is an even Fredholm module:

$$\operatorname{Index}_{(\mathcal{H},\rho,F)} \colon K_0(A) \to \mathbb{Z}, \quad [e] \mapsto \operatorname{ind}(\rho_-(e) F_+ \rho_+(e)),$$

where e is a projection, $\rho = \begin{pmatrix} \rho_+ & \\ & \rho_- \end{pmatrix}$ and $F = \begin{pmatrix} & F_+ \\ F_- & \end{pmatrix}$ with respect to the grading $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- If $(\mathcal{H}, \rho, F)$ is an odd Fredholm module:

$$\operatorname{Index}_{(\mathcal{H},\rho,F)} \colon K_1(A) \to \mathbb{Z}, \quad [u] \mapsto \operatorname{ind}(P \rho(u) P),$$

where $P := \frac{F+1}{2}$.

How do you compute these indices? Connes discovered a very useful formula, but we need some smoothness/summability conditions.

10.2 Finite summability in K-homology

10.2.1 Smooth extensions (Douglas, 1980s)

Douglas defined fine analytic properties of extensions. Recall that

Definition 10.8. Let $\mathcal{H}$ be a separable Hilbert space. The *Schatten p-ideal* is the two-sided (non-closed) ideal

$$\mathcal{L}^p(\mathcal{H}) := \{T \in \mathbb{K}(\mathcal{H}) \mid (\mathrm{s}_n(T))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})\},$$

where $\mathrm{s}_n(T)$'s denote the singular values of T.

Definition 10.9. Let $\mathcal{A}$ be a dense *-subalgebra of a C^*-algebra A. We say that an extension $\tau\colon A \to Q(\mathcal{H})$ is *p-smooth* on $\mathcal{A}$, if there exists a linear map $\eta\colon A \to \mathbb{B}(\mathcal{H})$, such that

$$\eta(ab) - \eta(a)\eta(b) \in \mathcal{L}^p(\mathcal{H}), \quad \eta(a^*) - \eta(a) \in \mathcal{L}^p(\mathcal{H}) \qquad \text{for all } a, b \in \mathcal{A},$$

and

$$\tau(a) = \eta(a) + \mathbb{K}(\mathcal{H}) \qquad \text{for all } a \in \mathcal{A}.$$

Example 10.10. Let X be a compact metrisable space, $\tau\colon \mathrm{C}(X) \to Q(\mathcal{H})$. Let $X \hookrightarrow \mathbb{C}^n$ be an embedding and restrict the coordinate functions $\{z_i\}_{i=1}^n$ to X. Then τ is p-smooth if there exists $\{T_i\}_{i=1}^n \subseteq \mathbb{B}(\mathcal{H})$, such that $\tau(z_i) = T_i + \mathbb{K}(\mathcal{H})$ and that $[T_i, T_j] \in \mathcal{L}^p(\mathcal{H})$, $[T_i, T_i^*] \in \mathcal{L}^p(\mathcal{H})$.

Theorem 10.11 (Douglas–Voiculescu)**.** *If $n \geq 2$. Then every $(n-1)$-smooth extension of* $\mathrm{C}(\mathbb{S}^{2n-1})$ *is trivial, and there exists non-trivial p-smooth extensions for $p > n$.*

10.2.2 Finite summable Fredholm modules (1980s)

Definition 10.12. Let $(\mathcal{H}, \rho\colon A \to \mathbb{B}(\mathcal{H}), F)$ be an normalised (i.e. $F^2 = 1$ and $F = F^*$) even or odd Fredholm module. It is *p-summable* on a dense *-subalgebra $\mathcal{A}$ if $[F, \rho(a)] \in \mathcal{L}^p(\mathcal{H})$ for all $a \in \mathcal{A}$.

Remark 10.13. There are several "layers" of representatives of K-homology classes:

$$\text{Spectral triples} \Rightarrow \text{Fredholm modules} \Rightarrow \text{Extensions}$$

Each higher layer representative with fine analytic properties (smoothness) gives rise to some analytic properties to the lower level representatives:

Theorem 10.14. *Let $(\mathcal{H}, \rho, F)$ be a p-summable spectral triple on $\mathcal{A} \subseteq A$. Then there is a $p/2$-smooth extension $\tau\colon A \to Q(\mathcal{H})$ on $\mathcal{A}$, defined by*

$$\tau\colon A \to Q(\mathcal{H}), \quad \tau(a) := P\rho(a)P$$
$$\eta\colon \mathcal{A} \to \mathbb{B}(\mathcal{H}), \quad \eta(a) := P\rho(a)P.$$

where $P := \frac{F+1}{2}$.

Proof. Notice that

$$P\rho(ab)P - P\rho(a)P^2\rho(b)P = -P[P, \rho(a)][P, \rho(b)] \subseteq \mathcal{L}^p \cdot \mathcal{L}^p \subseteq \mathcal{L}^{p/2}. \qquad \square$$

This cannot, however, be reversed: it is sometimes not even possible to find, e.g. spectral triples with smoothness conditions, out of a summable Fredholm module.

Theorem 10.15 (Connes' index formula)**.** *Let $(\mathcal{H}, \rho, F)$ be an even or odd Fredholm module over a* C^**-algebra A, which is p-summable on a dense *-subalgebra $\mathcal{A} \subseteq A$ satisfying* $\mathrm{K}_*(\mathcal{A}) \cong \mathrm{K}_*(A)$ *(e.g. if $\mathcal{A}$ is closed under holomorphic functional calculus). Then the index map* $\mathrm{Index}_{(\mathcal{H},\rho,F)}$ *is given by the following formulas:*

- *If $(\mathcal{H}, \rho, F)$ is even:*

$$\operatorname{Index}_{(\mathcal{H},\rho,F)}([e]) = a_n \operatorname{str}(e[F,e]^n), \quad \textit{for } e \textit{ a projection in } \mathcal{A}.$$

- *If $(\mathcal{H}, \rho, F)$ is odd:*

$$\operatorname{Index}_{(\mathcal{H},\rho,F)}([u]) = b_n \operatorname{tr}(u^*([F,u][F,u^*])^n [F,u]), \quad \textit{for } u \textit{ a unitary in } \mathcal{A}.$$

Here n is any even number, which is large enough such that $e[F,e]^n$ (or $u^*([F,u][F,u^*])^n[F,u]$) is trace-class; a_n and b_n are constants depending only on n; tr is the trace on $\mathbb{B}(\mathcal{H})$ and str is the supertrace on the $\mathbb{Z}/2$-graded $\mathbb{B}(\mathcal{H})$.

Example 10.16 (Toeplitz index theorem). Let $A = \mathrm{C}(\mathbb{T})$. Then A is represented on $\mathcal{H} = L^2(\mathbb{T})$ via multiplication. Let $P\colon \mathcal{H} \to \mathcal{H}$ be the projection onto the Hardy space

$$\mathrm{H}^2(\mathbb{T}) := \overline{\operatorname{span}}\{z^n \mid n \geq 0\}$$

and define $F := 2P - 1$.

Let $f \in \mathrm{C}(\mathbb{T})$. Then:

- $[F, f]$ has finite rank iff f is a trigonometric polynomial.
- For any $p > 1$: $[F, f] \in \mathcal{L}^p(\mathcal{H})$ iff $f \in \mathrm{C}^\infty(\mathbb{T})$.

Let $\mathcal{A} := \mathrm{C}^\infty(\mathbb{T})$. Then the Fredholm module $(\mathcal{H}, \rho, F)$ is p-summable on $\mathcal{A}$ for any $p > 1$. Connes' index formula yields

$$\operatorname{Index}_{(\mathcal{H},\rho,F)}([u]) = -\frac{1}{2\pi i}\int u^{-1}\,\mathrm{d}u = -\operatorname{wind}(u),$$

for $u \in \mathrm{C}^\infty(\mathbb{T})$ unitary. This is the Toeplitz index theorem.

Example 10.17 (Reduced group C*-algebra of free group). Let $A := \mathrm{C}^*_\mathrm{r}(\mathbb{F}_2)$ be the reduced group C*-algebra of $\mathbb{F}_2$, the free group of two generators. This is the closed linear span (closed under the operator norm) of $\{\delta_g \mid g \in \mathbb{F}_2\}$. Let T be the graph whose vertices are $T^0 = \mathbb{F}_2$ and whose edges T^1 are such that there is a unique edge between g and h iff gh^{-1} is a generator or its inverse. Then $\mathbb{F}_2$ acts on T. This induces an action

$$\rho\colon A \to \mathbb{B}(\mathcal{H}), \qquad \text{where } \mathcal{H} := \ell^2(T^0) \oplus \ell^2(T^1).$$

Define

$$U\colon \ell^2(T^0) \to \ell^2(T^1), \qquad U(\delta_v) := \begin{cases} 0 & v = 1, \\ \delta_{\ell(v)} & \text{otherwise,} \end{cases}$$

where $\ell(v)$ is the unique edge connecting v to the unique edge closet to the neutral element $1 \in \mathbb{F}_2$.

Define

$$F := \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Then $F = F^*$. $F^2 - 1$ has rank 1. $[F, \rho(\delta_g)]$ has finite rank for all g because $\delta_g U \delta_{g^{-1}} - U$ has finite rank.

Let $\mathcal{A}$ be the closure of $\mathbb{C}\mathbb{F}_2$ under holomorphic functional calculus. Then $\mathcal{A}$ is dense in A because it contains a dense subset $\mathbb{C}\mathbb{F}_2$. And $(\mathcal{H}, \rho, F)$ is 1-summable on $\mathcal{A}$.

Lemma 10.18. *Let $P \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Q \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ be bounded operators between Hilbert spaces such that $1 - PQ$ and $1 - QP$ are trace class. Then*

$$\operatorname{ind}(P) = \operatorname{tr}(1 - QP) - \operatorname{tr}(1 - PQ).$$

Theorem 10.19 (Kaplansky conjecture for $\mathbb{F}_2$). *There are no non-trivial projections in $\mathrm{C}^*_\mathrm{r}(\mathbb{F}_2)$.*

Proof by Connes–Cuntz. Let $(\mathcal{H}, \rho, F)$ be the normalisation of the Fredholm module as above, that is, $F^2 = 1$. It is 1-summable on $\mathcal{A}$. By Connes' index formula, we have

$$\mathrm{Index}_{(\mathcal{H},\rho,F)}([e]) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}1 & \\ & -1\end{pmatrix} F[F, \rho(e)]\right), \qquad \text{if } e \text{ is a projection in } \mathcal{A}.$$

Let $\tau\colon C^*_r(\mathbb{F}_2) \to \mathbb{C}$ be the canonical trace, i.e. $\tau(\delta_1) = 1$ and $\tau(\delta_g) = 0$ for $g \neq 1$. It is faithful and positive. In particular, we have

$$\tau(a) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}1 & \\ & -1\end{pmatrix} F[F, \rho(a)]\right), \qquad \text{if } a \in \mathcal{A}.$$

Now let $e \in A$ be a projection. It is unitarily equivalent to a projection $e' \in \mathcal{A}$ because the inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism $K_0(\mathcal{A}) \xrightarrow{\cong} K_0(A)$ (since $\mathcal{A}$ is stable under holomorphic calculus). Then

$$\tau(e) = \tau(ue'u^*) = \tau(e') = \mathrm{Index}_{(\mathcal{H},\rho,F)}([e'])$$

must be an integer. Since e is a projection, we have $\tau(e) \in [0, 1]$. Therefore $\tau(e) = 1$ or $\tau(e) = 0$.

If $\tau(e) = 0$. Then $\tau(e^*e) = \tau(e^2) = \tau(e) = 0$. Since τ is faithful, this means $e^*e = 0$, so $e = 0$. If $\tau(e) = 1$. Then $\tau(1 - e) = \tau(1) - \tau(e) = 0$ and $1 - e$ is also a projection. Then $1 - e = 0$ and $e = 1$. Above all, $e = 0$ or $e = 1$. □

June 7, 2022

E-theory

Speaker: Mick Gielen (Radboud University Nijmegen)

Throughout this section, we shall always work with separable C*-algebras.

Motivation Recall that the canonical functor $C^*\mathsf{Sep} \to \mathsf{KK}$ is the universal split-exact, homotopy-invariant, stable functor. This means if $F\colon C^*\mathsf{Sep} \to \mathsf{C}$ is a functor into an additive category C which satisfies the same properties, then F factors uniquely through KK.

If one replaces "split-exact" by "half-exact", then the universal functor is another canonical functor $C^*\mathsf{Sep} \to \mathsf{E}$, where E is the E-theory category. The composition of arrows in this category is an analog of the Kasparov product, but has an easier formulation.

11.1 Asymptotic morphisms

Definition 11.1. Let A and B be C*-algebras. An *asymptotic morphism* from A to B is a family of maps (not even necessarily linear)

$$\{\phi_t\colon A \to B\}_{t\in[1,\infty)}$$

such that:

- For all $a \in A$, $t \mapsto \phi_t(a)$ is continuous.
- $\{\phi_t\}$ is "asymptotically" *-linear and multiplicative. That is,

$$\lim_{t\to\infty} \|\phi_t(a+b) - \phi_t(a) - \phi_t(b)\| \to 0, \quad \lim_{t\to\infty} \|\phi_t(a^*) - \phi_t(a)^*\| \to 0, \quad \lim_{t\to\infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| \to 0.$$

Definition 11.2. A *homotopy* between asymptotic morphisms $\{\phi^0_t\colon A \to B\}_t$ and $\{\phi^1_t\colon A \to B\}_t$ is an asymptotic morphism

$$\{\Phi_t\colon A \to IB\}_{t\in[1,\infty)}$$

such that $\mathrm{ev}_t \circ \Phi_t = \phi^i_t$ for $i = 0, 1$ and $t \in [1, \infty)$.

We write $[\![A, B]\!]$ for the set of homotopy classes of asymptotic morphisms A to B, and $\langle\phi_t\rangle_{t\in[1,\infty)}$ for homotopy class of an asymptotic morphism $\{\phi_t\}_{t\in[1,\infty)}$. We shall omit the subscript $t \in [1, \infty)$ occasionally.

Notice that we have a map

$$[A, B] \to [\![A, B]\!], \qquad [\phi] \mapsto \langle \phi_t := \phi\colon A \to B \rangle_t$$

where $[A, B]$ denotes the homotopy classes of *-homomorphisms A to B. $[\phi]$ is the homotopy class of ϕ in $[A, B]$. $\langle \phi_t \rangle$ is the homotopy class of $\{\phi_t\}$ in $[\![A, B]\!]$.

Definition 11.3. Two asymptotic morphisms $\{\phi_t\colon A \to B\}$ and $\{\psi_t\colon A \to B\}$ are called *equivalent*, if

$$\lim_{t\to\infty} \|\phi_t(a) - \psi_t(a)\| = 0, \qquad \text{for all } a \in A.$$

In this case we have $[\phi] = [\psi] \in [\![A, B]\!]$.

Proposition 11.4. *Let $\{\phi_t\colon A \to B\}$ be an asymptotic morphism. Then*

$$\limsup_{t\to\infty} \|\phi_t(a)\| \le \|a\|, \qquad \textit{for any } a \in A$$

i.e. $\{\phi_t\}$ is "asymptotically contractive".

So $\{\phi_t\}$ defines a map $\phi\colon A \to \mathrm{C_b}([1, \infty), B)$, which induces a *-homomorphism

$$\phi\colon A \to B_\infty := \mathrm{C_b}([1, \infty), B)/\mathrm{C_0}([1, \infty), B).$$

In particular, $\phi = \psi$ iff $\{\phi_t\}$ is equivalent to $\{\psi_t\}$.

Conversely, if $\phi\colon A \to B_\infty$ is a *-homomorphism. Then it induces an asymptotic morphism $\{\phi_t\colon A \to B\}$ by picking a set-theoretic lift $\phi\colon A \to \mathrm{C_b}([1, \infty), B)$ and define $\phi_t(a) := \phi(a)(t)$. Different lifts determines the same asymptotic morphism up to equivalence.

Definition 11.5. An asymptotic morphism $\{\phi_t\colon A \to B\}$ is *completely positive* (*cp* for short), if $\phi_t\colon A \to B$ is a completely positive and contractive *-linear map for all t.

We write $[\![A, B]\!]_{\mathrm{cp}}$ for the homotopy classes of *cp* asymptotic morphisms.

Example 11.6. Suppose A is nuclear and $\{\phi_t\colon A \to B\}$ is an asymptotic morphism. Then the induced *-homomorphism $\phi\colon A \to B_\infty$ lifts to a cpc map $A \to \mathrm{C_b}([1, \infty), B)$. Since different lifts differ by a map which is asymptotically vanishing, they define equivalent asymptotic morphisms. Therefore, $\{\phi_t\}$ is equivalent to a cp asymptotic morphism. So $[\![A, B]\!]_{\mathrm{cp}} = [\![A, B]\!]$ if A is nuclear.

11.1.1 Tensor product

If $\{\phi_t\colon A \to C\}$ and $\{\psi\colon B \to D\}$ are asymptotic morphisms. Then we can define their tensor product asymptotic morphism

$$\{\phi_t \otimes_{\max} \psi_t\colon A \otimes_{\max} B \to C \otimes_{\max} D\}.$$

Notice that this only applies to the *maximal* tensor product of C*-algebras.

In particular: we can define the suspension $\{\mathrm{S}\phi_t\colon \mathrm{S}A \to \mathrm{S}B\}$ of an asymptotic morphism $\{\phi_t\colon A \to B\}$.

11.1.2 Composition

Let $\{\phi_t\colon A \to B\}$ and $\{\psi_t\colon B \to C\}$ be asymptotic morphisms. In general $\{\psi_t \circ \phi_t\colon A \to C\}$ is *not* an asymptotic morphism (due to uniform convergence issues). But we have a simple solution.

Proposition 11.7. *Let $r\colon [1, \infty) \to [1, \infty)$ be a continuous, unbounded and increasing function. Then $\langle \psi_{r(t)} \rangle = \langle \psi_t \rangle \in [\![B, C]\!]$.*

If the function r increases "quickly enough". Then $\{\psi_{r(t)} \circ \phi_t\}$ is an asymptotic morphism, whose homotopy class does not depend on the choice of the function r. This gives a composition of maps

$$[\![A, B]\!] \times [\![B, C]\!] \to [\![A, C]\!].$$

11.1.3 Addition

Choose an isomorphism $\mathbb{K} \cong \mathbb{M}_2(\mathbb{K})$, we can define an addition on $[\![A, B \otimes \mathbb{K}]\!]$. Let $\{\phi_t : A \to B \otimes \mathbb{K}\}$ and $\{\psi_t : A \to B \otimes \mathbb{K}\}$ be asympototic morphisms. Define the following asymptotic morphism

$$\phi_t \oplus \psi_t : A \to B \otimes (\mathbb{K} \oplus \mathbb{K}) \hookrightarrow B \otimes \mathbb{M}_2(\mathbb{K}) \cong B \otimes \mathbb{K}$$

as their sum.

This turns $[\![A, B \otimes \mathbb{K}]\!]$ into a semigroup, but not necessarily a group. In order to achieve a group structure, we have to use suspensions to do some delooping.

Proposition 11.8. *$[\![A, \mathrm{S}B \otimes \mathbb{K}]\!]$ is a group.*

11.2 E-theory

Definition 11.9. Let A and B be C*-algebras.

- The E-theory group $\mathrm{E}(A, B)$ is defined as

$$\mathrm{E}(A, B) := [\![\mathrm{S}A \otimes \mathbb{K}, \mathrm{S}B \otimes \mathbb{K}]\!].$$

- This gives an additive category E, whose
  - Objects are separable C*-algebras.
  - Arrows are elements in E-theory groups.
- There is a cp variant of E-theory:

$$\mathrm{E}_{\mathrm{cp}}(A, B) := [\![\mathrm{S}A \otimes \mathbb{K}, \mathrm{S}B \otimes \mathbb{K}]\!]_{\mathrm{cp}},$$

  which will be shown to agree with KK-theory.

Let

$$J \rightarrowtail A \overset{q}{\twoheadrightarrow} B \tag{$*$}$$

be a short exact sequence of C*-algebras. Let $\{u_t\}_{t\in[1,\infty)}$ be a continuous increasing approximate unit in J, which is *quasi-central* for A: that is, u_t asymptotically commmutes with A.

1. Let $\sigma : B \to A$ be a *-linear splitting. Then the collection

$$\phi_t : \mathrm{S}B \to J, \qquad \phi_t(f \otimes b) := f(u_t)\sigma(b)$$

   defines an asymptotic morphism, hence a class $\epsilon_q \in [\![\mathrm{S}B, J]\!]$, which does not depend on $\{u_t\}$ and σ. This ϵ_q is called the *connecting morphism.*
2. If ($*$) is cpc-split. Then we can choose the σ in 1 to be cpc. Then this defines an element

$$\epsilon_q \in [\![\mathrm{S}B, J]\!]_{\mathrm{cp}}.$$

3. If ($*$) splits, then $\epsilon_q = 0$. In this case we can consider the C*-algebra $E \subseteq \mathrm{I}A$ generated by $\mathrm{S}J$ and $\{\tau(a) \mid a \in A\} \subseteq \mathrm{I}A$, where $\tau(a)(s) := (1 - s)a + s\sigma(q(a))$. Then we get an extension

$$\mathrm{S}J \rightarrowtail E \twoheadrightarrow A$$

   whose connecting morphism $\eta_q \in [\![\mathrm{S}A, \mathrm{S}J]\!]$ is called the *splitting morphism.*

Proposition 11.10. *$\eta_q \oplus \mathrm{S}q$ defines an isomorphism in $\mathrm{E}(A, J \oplus B)$. So $\mathrm{C^*Sep} \to \mathsf{E}$ is split-exact.*

By construction, $\mathrm{C^*Sep} \to \mathsf{E}$ is automatically homotopy-invariant and stable. So it factors through the Kasparov category KK and there is a natural map $\mathrm{KK}(A, B) \to \mathrm{E}(A, B)$.

Corollary 11.11. E*-theory has Bott periodicity.*

Definition 11.12.

$$\mathrm{E}^0(A,B) := \mathrm{E}(A,B), \qquad \mathrm{E}^1(A,B) := \mathrm{E}(SA,B) \cong \mathrm{E}(A,SB).$$

Theorem 11.13. E*-theory is half-exact. i.e. for* any *extension $J \rightarrowtail A \twoheadrightarrow B$ of C^*-algebras and any C^*-algebra D, the sequences $\mathrm{E}(D,J) \to \mathrm{E}(D,A) \to \mathrm{E}(D,B)$ and $\mathrm{E}(B,D) \to \mathrm{E}(A,D) \to \mathrm{E}(J,D)$ are exact in the middle.*

Corollary 11.14. *For* any *extension $J \rightarrowtail A \twoheadrightarrow B$ of C^*-algebras, there is a six-term cyclic exact sequence*

$$\begin{array}{ccccc} \mathrm{E}^0(D,J) & \longrightarrow & \mathrm{E}^0(D,A) & \longrightarrow & \mathrm{E}^0(D,B) \\ \uparrow & & & & \downarrow \\ \mathrm{E}^1(D,B) & \longleftarrow & \mathrm{E}^1(D,A) & \longleftarrow & \mathrm{E}^1(D,J). \end{array}$$

Such a six-term exact sequence exists in E_{cp} if the extension $J \rightarrowtail A \twoheadrightarrow B$ is cpc-split. This is because the connecting morphism ϵ_q is required to be cp.

11.2.1 E-theory as a universal functor

Theorem 11.15. *The canonical functor $\mathrm{C}^*\mathsf{Sep} \to \mathsf{E}$ is the universal half-exact, homotopy-invariant and stable functor. That is, let $F\colon \mathrm{C}^*\mathsf{Sep} \to \mathsf{C}$ be a half-exact, homotopy-invariant and stable functor into an additive category C. Then it factors uniquely through $\mathrm{C}^*\mathsf{Sep} \to \mathsf{E}$.*

Sketch of proof. The most essential part is to construct the functor $\mathsf{E} \to \mathsf{C}$. Let ϕ be an asymptotic morphism which represents a class $\langle\phi\rangle \in [\![SA\otimes\mathbb{K}, SB\otimes\mathbb{K}]\!] = \mathrm{E}(A,B)$. We wish to define a morphism $F(\langle\phi\rangle)\colon F(A) \to F(B)$. We have $\langle S\phi\rangle \in [\![S^2A\otimes\mathbb{K}, S^2B\otimes\mathbb{K}]\!] = [\![A',B']\!]$ where $A' := S^2A\otimes\mathbb{K}$, $B' := S^2B\otimes\mathbb{K}$. It suffices to work with A' and B': any half-exact, homotopy-invariant and stable functor F satiesfies Bott periodicity (Theorem 2.7).

The asymptotic morphism $\langle S\phi\rangle \in [\![A',B']\!]$ induces a *-homomorphism $S\phi\colon A' \to B'_\infty$ where $B'_\infty := \mathrm{C_b}([1,\infty),B')/\mathrm{C_0}([1,\infty),B')$. Define D to be the pullback

$$\begin{array}{ccc} D & \longrightarrow & \mathrm{C_b}([1,\infty),B') \\ {\scriptstyle \pi_{A'}}\downarrow & & \downarrow \\ A' & \xrightarrow{S\phi} & B'_\infty. \end{array}$$

There is an extension of C^*-algebras

$$\mathrm{C_0}([1,\infty),B') \rightarrowtail D \overset{\pi_{A'}}{\twoheadrightarrow} A'$$

which is cpc-split if ϕ is cp.

Let $\rho_t := D \to \mathrm{C_b}([1,\infty),B') \xrightarrow{\mathrm{ev}_t} B'$. Then $\{\rho_t\}$ defines an asymptotic morphism from D to B', which is homotopic to $\{\rho_1\}$ in an obvious way. It induces the *-homomorphism $D \to \mathrm{C_b}([1,\infty),B') \to B'_\infty$, which equals $S\phi \circ \pi_{A'}$ because the pullback diagram commutes. Therefore,

$$\langle\rho_1\rangle = \langle\rho_t\rangle = \langle S\phi\circ\pi_{A'}\rangle = \langle S\phi\rangle \circ \langle\pi_{A'}\rangle \in [\![D,B']\!].$$

But $\mathrm{C_0}([1,\infty),B')$ is contractible! Since F is half-exact, homotopy-invariant and stable functor, we have a long exact sequence in F, which implies that $F(\pi_{A'})$ is invertible. Then we may define

$$F(\langle\phi\rangle) := F(\rho_1)\circ F(\pi_{A'})^{-1}\colon F(A') \to F(B').$$

This is the desired map $\mathrm{E}(A,B) \to \mathrm{Hom}(F(A'),F(B')) = \mathrm{Hom}(F(A),F(B))$. □

Remark 11.16. If we work instead in $\mathrm{E_{cp}}$, then we shall start with a cp asymptotic morphism ϕ from $SA \otimes \mathbb{K}$ to $SB \otimes \mathbb{K}$. Then the extension $C_0([1,\infty), B') \rightarrowtail D \overset{\pi_{A'}}{\twoheadrightarrow} A'$ is cpc-split and hence induces a long exact sequence in KK-theory. Then $[\pi_{A'}] \in \mathrm{KK}(D, A')$ is also invertible and the construction above gives a map $\mathrm{E_{cp}}(A,B) \to \mathrm{KK}(A,B)$. Since $\mathrm{E_{cp}}$ is also split-exact, by universality of KK-theory we obtain another factorisation $\mathrm{KK}(A,B) \to \mathrm{E_{cp}}(A,B)$. These two maps are easily seen to be the inverse to each other. Hence we conclude that:

Corollary 11.17.
- $\mathrm{E_{cp}}(A,B) \cong \mathrm{KK}(A,B)$ *for all* A, B.
- *If* A *is nuclear. Then* $[\![A,B]\!] = [\![A,B]\!]_{\mathrm{cp}}$ *(Example 11.6) and hence* $\mathrm{E}(A,B) \cong \mathrm{KK}(A,B)$.

June 14, 2022

K-theory of graph C*-algebras

Speaker: Yufan Ge (Leiden University)

Graphs are a class of interesting mathematical objects which has been studied for a long time. The idea of graph C*-algebras can be dated back to the theory of quiver algebras, in which the vertices are labelled by vector spaces and the edges are labelled by linear maps. In the situation of graph C*-algebras, they are replaced by Hilbert spaces and partial isometries.

In this section, we assume that all graphs are *directed*. The main reference for this section is [23].

12.1 Graph C*-algebras

Definition 12.1. A graph is a quadruple $E = (E^0, E^1, r, s)$, where E^0 and E^1 are two sets and $r, s\colon E^1 \rightrightarrows E^0$ are maps between them. An element in E^0 is called a *vertice* and an element in E^1 is called an *edge*. The maps r and s are called the *range* and *source* maps.

Let $e \in E^1$ with $r(e) = v$ and $s(e) = w$. We say v is the range of e and w is the source of e. We also say v receives an arrow e from w.

If $e \in E^1$ satisfies $r(e) = s(e)$. Then we say e is a *loop*.

A *path* is a word of edges $e_1 e_2 \ldots e_n$ where $s(e_i) = r(e_{i+1})$. The *length* of a path is the length of the word. We write E^i for the set of paths of length i. Clearly, we may view vertices as length-zero paths and edges as length-one paths.

A *cycle* is a path $e_1 \ldots e_n$ such that $r(e_1) = s(e_n)$.

Example 12.2. Consider the graph E_n with a unique vertice and n edges (so each of them must be a loop at the unique vertice). The corresponding graph C*-algebra is the Cuntz algebra $\mathcal{O}_n$.

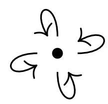

Figure 12.1: The graph E_4.

Definition 12.3. Let E be a graph with $E^0 = \{v_1, \ldots, v_n\}$. The *adjacency matrix* A_E is defined as the $n \times n$-matrix whose (i,j)-entry is

$$(A_E)_{i,j} := \#\{e \in E^1 \mid s(e) = e_j,\ r(e) = e_i\}.$$

Definition 12.4. Let E be a row-finite graph. That is, every vertice receives finite many edges. A Cuntz–Krieger E-family consists of two subsets

$$S := \{S_e \mid e \in E^1\}, \qquad P := \{P_v \mid v \in E^0\}$$

in a C*-algebra A, such that every S_e is a partial isometry and every P_v is a projection, and satisfy the following Cuntz–Krieger relations:

(CK1) $S_e^* S_e = P_{s(e)}$.

(CK2) $\sum_{r(e)=v} S_e S_e^* = P_v$.

We write $\mathrm{C}^*(S, P)$ for the C^*-algebra generated by S and P; let us call it the *Cuntz–Krieger algebra.*

Proposition 12.5. *Let E be a row-finite graph. Let $\{S, P\}$ be a Cuntz–Krieger E-family. Then*

- *$\{S_e S_e^* \mid e \in E^1\}$ are mutally orthogonal.*
- *$S_e^* S_f \neq 0$ iff $e = f$.*
- *If $S_e S_f \neq 0$, then $s(e) = r(f)$.*
- *If $S_e S_f^* \neq 0$, then $s(e) = s(f)$.*

Corollary 12.6. *If $\{S, P\}$ is a Cuntz–Krieger E-family. Then*

$$\mathrm{C}^*(S, P) = \overline{\operatorname{span}} \left\{ S_\mu S_\nu^* \,\middle|\, \mu, \nu \in \bigcup_{i=0}^{\infty} E^i \right\},$$

where

$$S_\mu := S_{e_1} \dots S_{e_n} \qquad \text{where } \mu = e_1 \dots e_n.$$

Example 12.7. Consider the graph as in Figure 12.2. The Cuntz–Krieger relations are

$$P_v + P_w = \mathrm{id}, \quad S_e^* S_e = P_v, \quad S_f^* S_f = P_v$$
$$S_f S_f^* = P_w, \quad S_e S_f = S_e^* S_f = 0.$$

We may choose any representation of the generators: different representations yield isomorphic Cuntz–Krieger algebras, which are all isomorphic to the Toeplitz algebra $\mathcal{T}$. In fact, using the Cuntz–Krieger relations one can check that $T := S_e + S_f$ is an isometry, and

$$P_v = TT^*, \quad P_w = \mathrm{id} - TT^*, \quad S_e = TP_e, \quad S_f = TP_f.$$

Therefore, $\mathrm{C}^*(S, P)$ is the universal C^*-algebra generated by an isometry, which is just the Toeplitz algebra by Coburn's Theorem.

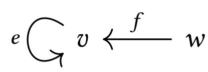

Figure 12.2: A graph: its graph C^*-algebra and Cuntz–Krieger algebras are all $\mathcal{T}$.

Proposition 12.8. *Let E be a row-finite graph. There exists a universal Cuntz–Krieger algebra generated by a Cuntz–Krieger E-family $\{S, P\}$: that is, for any other Cuntz–Krieger algebra which is generated by another Cuntz–Krieger E-family $\{T, Q\}$, there exists a *-homomorphism π with*

$$\pi(S_e) = T_e, \ \pi(P_v) = Q_v, \qquad \text{for all } e \in E^1 \text{ and } v \in E^0.$$

Definition 12.9. The universal C^*-algebra $\mathrm{C}^*(E)$ defined in the previous proposition is called the *graph C^*-algebra* of the graph E.

Theorem 12.10 (Gauge-invariant uniqueness theorem)**.** *Let E be a row-finite graph and $\{S, P\}$ is a Cuntz–Krieger E-family in a C^*-algebra B, such that $Q_v \neq 0$ for all $v \in E^0$. If there exists a continuous action $\beta \colon \mathbb{T} \to \mathrm{Aut}(B)$ such that*

$$\beta_z(S_e) = zS_e, \qquad \beta_z(P_v) = P_v$$

for all e and v. Then $\mathrm{C}^*(E) \cong \mathrm{C}^*(S, P)$.

Sketch of the proof.

- $C^*(E)$ carries a canonical gauge action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(C^*(E))$.
- Let
$$\Phi\colon C^*(E) \to C^*(E), \qquad a \mapsto \int_{\mathbb{T}} \gamma_z(a)\,\mathrm{d}z.$$
Then the image of Φ is the γ-fixed point algebra
$$C^*(E)^\gamma := \{a \in C^*(E) \mid \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}.$$
Moreover, Φ is contractive and faithful (i.e. $\Phi(a^*a) = 0$ iff $a = 0$).
- A lemma:

  Lemma 12.11. *The map $\pi\colon C^*(E) \to C^*(S,P)$ is injective on $C^*(E)^\gamma$.*

- Clearly $\pi\colon C^*(E) \to C^*(S,P)$ is surjective. Notice that
$$\|\pi(\Phi(a))\| \le \int_{\mathbb{T}} \|\pi(\gamma_z(a))\|\,\mathrm{d}z = \int_{\mathbb{T}} \|\beta_z(\pi(a))\|\,\mathrm{d}z = \|\pi(a)\|.$$
If $\pi(a) = 0$. Then $\pi(a^*a) = 0$. So $\pi(\Phi(a^*a)) = 0$. But $\Phi(a^*a) \in C^*(E)^\gamma$ and π is injective on it, we have $\Phi(a^*a) = 0$. Then $a = 0$. So π is injective on the whole of $C^*(E)$, which implies that $\pi\colon C^*(E) \to C^*(S,P)$ is an isomorphism. □

Theorem 12.12 (Cuntz–Krieger uniqueness theorem)**.** *Let E be a row-finite graph such that every cycle has an entry. (That is, given any cycle $e_1 \dots e_n$, there exists an edge e with $e \neq e_i$ for all $i = 1, \dots n$, and that $r(e) = r(e_i)$ for some e_i). Then the Cuntz–Krieger algebra $C^*(S,P)$ generated by any Cuntz–Krieger E-family $\{S,P\}$ is isomorphic to the graph C^*-algebra $C^*(E)$.*

12.2 K-theory of graph C*-algebras

Graph C^*-algebras have many projections which represent classes in K_0. Due to the Cuntz–Krieger relations, these classes also satisfy certain relations, allowing us to compute the K-theory. Recall that two projections p and q represent the same class if $p = u^*u$ and $q = uu^*$. Using (CK2):
$$[P_v] = \left[\sum_{r(e)=v} S_e S_e^*\right] = \sum_{r(e)=v} [S_e^* S_e] = \sum_{r(e)=v} [P_{s(e)}] = \sum_{w \in E^0} A_E(v,w)[P_w].$$

We will prove the following main theorem:

Theorem 12.13. *Let E be a row-finite graph without* sources*. (That is, for any $v \in E^0$, there exists $e \in E^1$ with $r(e) = v$). Then*
$$K_0(C^*(E)) \cong \mathrm{coker}(1 - A_E^{\mathrm{t}}), \qquad K_1(C^*(E)) \cong \ker(1 - A_E^{\mathrm{t}}).$$
Here we view the adjacency matrix A_E as a linear map $\mathbb{Z}^{|E^0|} \to \mathbb{Z}^{|E^0|}$.

Example 12.14. Consider the graph E_n as in Example 12.2. The graph C^*-algebra is the Cuntz algebra $\mathcal{O}_n$. Applying the theorem we have:
$$K_0(\mathcal{O}_n) = \mathrm{coker}(1-n) = \mathbb{Z}/(n-1), \qquad K_1(\mathcal{O}_n) = \ker(1-n) = 0.$$

The proof of the theorem is based on the dual Pimsner–Voiculescu sequence, and the K-theory of AF-algebras.

12.2.1 Dual Pimsner–Voiculescu sequence

Recall that an AF-algebra is the direct limit of an increasing sequence of finite-dimensional C*-algebras $A = \overline{\bigcup_{i=1}^{\infty} A_n}$. Any finite-dimensional C*-algebra is the direct sum of some finite-dimensional matrix algebras $\bigoplus_{i=1}^{n} \mathbb{M}_{d_i}(\mathbb{C})$. Since K_i's are covariant functors which preserves direct sums and direct limits. We have

$$\mathrm{K}_0(A) = \lim_{n} \mathbb{Z}^{\sum_i d_i}, \qquad \mathrm{K}_1(A) = 0.$$

Recall the definition of crossed products:

Definition 12.15. A crossed product for a C*-dynamical system (A, G, α) is a C*-algebra $A \rtimes_\alpha G$ together with *-homomorphisms

$$i_A \colon A \to \mathcal{M}(A \rtimes_\alpha G), \quad i_G \colon G \to \mathcal{UM}(A \rtimes_\alpha G),$$

characterised uniquely by the following properties:

- $i_A(\alpha_s(a)) = i_G(s) i_A(a) i_G(s)^*$ for all $a \in A$ and $s \in G$.
- *Universality.* If (π, U) is a covariant representation of (A, G, α) on a Hilbert space $\mathcal{H}$. Then there exists a *-homomorphism $\pi \rtimes U \colon \mathcal{M}(A \rtimes_\alpha G) \to \mathbb{B}(\mathcal{H})$ such that

$$\pi \rtimes U \circ i_A = \pi, \qquad \pi \rtimes U \circ i_G = U.$$

- The linear span

$$\operatorname{span}\{ i_A(a) i_G(z) \mid z \in \mathrm{C_c}(G)\},$$

is closed in $A \rtimes_\alpha G$. where

$$i_G(z) := \int_G z(s) i_G(s)\, \mathrm{d}z.$$

Let (A, G, α) be a C*-dynamical system with G abelian. There is a dual action $\hat{\alpha}$ of $\hat{G}$ on $A \rtimes_\alpha G$ by

$$\hat{\alpha}_\gamma(i_A(a)) := i_A(a), \qquad \hat{\alpha}_\gamma(i_G(z)) := i_G(\gamma(z)).$$

Theorem 12.16 (Takai duality)**.** *Let (A, G, α) be a* C*-*dynamical system with G abelian. Then*

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathbb{K}(L^2(G)).$$

Apply Pimsner–Voiculescu sequence to the C*-algebra $\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$ with the $\mathbb{Z}$-action $\hat{\gamma}$, and identify

$$\mathrm{K}_i((\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}) \rtimes_{\hat{\gamma}^{-1}} \mathbb{Z}) \cong \mathrm{K}_i((\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}) \rtimes_{\hat{\gamma}} \mathbb{Z}) \cong \mathrm{K}_i(\mathrm{C}^*(E) \otimes \mathbb{K}(L^2(G))) \cong \mathrm{K}_i(\mathrm{C}^*(E))^{6},$$

we obtain the following *dual Pimsner–Voiculescu sequence*:

$$\begin{array}{ccccc}
\mathrm{K}_0(\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}) & \xrightarrow{\mathrm{id} - \hat{\gamma}_*^{-1}} & \mathrm{K}_0(\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}) & \longrightarrow & \mathrm{K}_0(\mathrm{C}^*(E)) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(\mathrm{C}^*(E)) & \longleftarrow & \mathrm{K}_1(\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}) & \xleftarrow{\mathrm{id} - \hat{\gamma}_*^{-1}} & \mathrm{K}_1(\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}),
\end{array}$$

where $\hat{\gamma}$ is the dual action of the gauge action and $\hat{\gamma}_*$ is the induced map in K-theory.

[6] Here we use the fact that $A \rtimes_\alpha G \cong A \rtimes_{\alpha^{-1}} G$. This is because a G-action α on A is equivalent to a G^{op}-action α^{-1} on A; but $G^{\mathrm{op}} \cong G$.

12.2.2 Construction of the graph $E \rtimes_1 \mathbb{Z}$

Our next goal is construct another graph $E \rtimes_1 \mathbb{Z}$ such that:

> The graph C^*-algebra $\mathrm{C}^*(E \rtimes_1 \mathbb{Z})$ is AF and isomorphic to the crossed product $\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$.

As such is achieved, since AF-algebras have trivial K_1, the cyclic sequence breaks down to

$$0 \to \mathrm{K}_1(\mathrm{C}^*(E)) \to \mathrm{K}_0(\mathrm{C}^*(E) \rtimes_\alpha \mathbb{T}) \xrightarrow{\mathrm{id} - \hat{\gamma}_*^{-1}} \mathrm{K}_0(\mathrm{C}^*(E) \rtimes_\alpha \mathbb{T}) \to \mathrm{K}_0(\mathrm{C}^*(E)) \to 0$$

So

$$\mathrm{K}_0(\mathrm{C}^*(E)) \cong \operatorname{coker}(\mathrm{id} - \hat{\gamma}_*^{-1}), \qquad \mathrm{K}_1(\mathrm{C}^*(E)) \cong \ker(\mathrm{id} - \hat{\gamma}_*^{-1}).$$

Then we may identify $\hat{\gamma}$ with A_E^{t} to conclude the main Theorem 12.13.

Definition 12.17. Define the graph $E \rtimes_1 \mathbb{Z}$ by

- $(E \rtimes_1 \mathbb{Z})^i := E^i \times \mathbb{Z}$.
- $r(e, n) := (r(e), n - 1)$.
- $s(e, n) := (s(e), n)$.

See Figure 12.3 for an example.

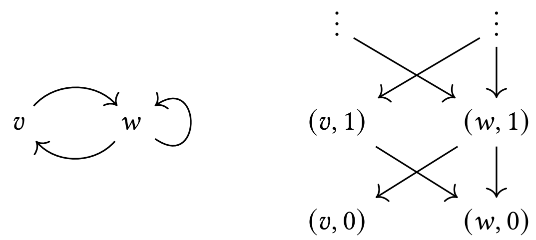

Figure 12.3: An example of E and $E \rtimes_1 \mathbb{Z}$.

Remark 12.18.

- It is easy to show that there are no cycles in $E \rtimes_1 \mathbb{Z}$.
- $E \rtimes_1 \mathbb{Z}$ carries an action of $\mathbb{Z}$:
$$\beta_m(e, n) = (e, n + m).$$
This induces an action
$$\beta \colon \hat{\mathbb{T}} = \mathbb{Z} \to \mathrm{Aut}(\mathrm{C}^*(E \rtimes_1 \mathbb{Z})).$$

Lemma 12.19. *There is an isomorphism* $\phi \colon \mathrm{C}^*(E \rtimes_1 \mathbb{Z}) \xrightarrow{\cong} \mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$ *intertwining* β *and* $\hat{\gamma}$.

Proof of Lemma 12.19. Define a Cuntz–Krieger $(E \rtimes_1 \mathbb{Z})$-family $\{T, Q\}$ in $\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$ by

$$T_{(e,n)} := i_{\mathrm{C}^*(E)}(S_e) i_{\mathbb{T}}(f_n), \qquad Q_{(v,n)} := i_{\mathrm{C}^*(E)}(P_v) i_{\mathbb{T}}(f_n),$$

where $f_n \in \mathrm{C}(\mathbb{T})$ is the function $z \mapsto z^n$. By universal property of the graph C^*-algebra $\mathrm{C}^*(E \rtimes_1 \mathbb{Z})$, there is a *-homomorphism $\phi \colon \mathrm{C}^*(E \rtimes_1 \mathbb{Z}) \to \mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$, whose image is the Cuntz–Pimsner algebra generated by $\{T, Q\}$. Since $E \rtimes_1 \mathbb{Z}$ has no cycle. By Cuntz–Krieger uniqueness theorem 12.12, the map ϕ is an isomorphism onto its image. But notice that f_n is dense in $\mathrm{C}(\mathbb{T})$. Therefore $i_{\mathrm{C}^*(E)}(a) i_{\mathbb{T}}(f_n)$ spans a dense subset of $\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$. Hence $\phi \colon \mathrm{C}^*(E \rtimes_1 \mathbb{Z}) \xrightarrow{\cong} \mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$ is an isomorphism. ◻

Proof of Main Theorem 12.13. Let F_n be the subgraph of $E \rtimes_1 \mathbb{Z}$ with

$$F_n^0 := \{(v,k) \in (E \rtimes_1 \mathbb{Z})^0 \mid k \leq n\}, \qquad F_n^1 := \{(v,k) \in (E \rtimes_1 \mathbb{Z})^1 \mid k \leq n\}.$$

Then $\mathrm{C}^*(F_n)$ is finite. In particular, $\mathrm{K}_0(\mathrm{C}^*(F_n))$ is the free abelian group $\mathbb{Z}^{|F_n^0|}$ generated by $P_{(v,n)}$ (c.f. [23, Lemma 7.13]). The graph C^*-algebra $\mathrm{C}^*(E \rtimes_1 \mathbb{Z})$ is the direct limit of $\mathrm{C}^*(F_k)$, hence AF.

It remains to identify $\hat{\gamma}_*^{-1}$, or equivalently β^{-1} (by Lemma 12.19), with A_E^{t}. Using the Cuntz–Krieger relations, we have

$$\begin{aligned}
\beta^{-1}[P_{(v,n)}] = [P_{(v,n-1)}] &= \left[\sum_{r(e,k)=(v,n-1)} S_{(e,k)}S_{(e,k)}^*\right] \\
&= \sum_{r(e)=v} \left[S_{(e,n)}S_{(e,n)}^*\right] \\
&= \sum_{r(e)=v} \left[S_{(e,n)}^*S_{(e,n)}\right] \\
&= \sum_{r(e)=v} [P_{(e,n)}] \\
&= \sum_{w\in E^0} A_E(v,w)[P_{(w,n)}].
\end{aligned}$$

□

June 22, 2022

K-theory of Cuntz–Pimsner algebras

Speaker: Francesca Arici (Leiden University)

References and historical remark:

- In [7]. Cuntz defined the well-known Cuntz algebras $\mathcal{O}_n$.
- In [10], Cuntz and Krieger defined graph C^*-algebras. Cuntz algebras are special cases of them.
- In [22], Pimsner constructed two C^*-algebras $\mathcal{T}_E$ and $\mathcal{O}_E$ from an injective C^*-correspondence (E,ϕ). The C^*-algebra $\mathcal{O}_E$, called the *Cuntz–Pimsner algebra* of (E,ϕ), generalises both Cuntz–Krieger algebras and crossed product by $\mathbb{Z}$.

The Cuntz–Pimsner algebras have the following properties:

- Similar to crossed products by $\mathbb{Z}$: the Cuntz–Pimsner algebras have a long exact sequence in KK-theory generalising the Pimsner–Voiculescu exact sequence.
- Similar to graph C^*-algebras: in nice cases, the Cuntz–Pimsner algebras are also universal and carry a gauge action.

13.1 Toeplitz–Pimsner algebras

Let (E,ϕ) be a *injective* C^*-correspondence over A. That is, E is a Hilbert A-module and $\phi\colon A \to \mathbb{B}_A(E)$ is a non-degenerate injective *-homomorphism. For today, we also assume that $\operatorname{im}\phi \subseteq \mathbb{K}_A(E)$. Then (E,ϕ) defines a Kasparov (A,A)-module.

Definition 13.1. Let

$$E^{(0)} := A, \quad E^{(k)} := \underbrace{E \otimes_\phi E \otimes_\phi \cdots \otimes_\phi E}_{k \text{ copies of } E}, \qquad \text{for all } k > 0.$$

The *Fock correspondence* is the C*-correspondence (E_+, ϕ_+) over A, where

$$E_+ := \bigoplus_{k\in\mathbb{N}} E^{(k)}, \quad \phi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) := \phi(a)\xi_1 \otimes \cdots \xi_n.$$

Using the Fock correspondence, we can construct an explicit representation of the C*-algebra $\mathcal{T}_E$.

Definition 13.2. Let $\xi \in E$. The *Toeplitz operator* (or *shift operator*, or *creation operator*) associated to ξ, is the bounded operator T_ξ on E_+, defined by

$$T_\xi(\eta_1 \otimes \cdots \otimes \eta_n) := \xi \otimes \eta_1 \otimes \cdots \otimes \eta_n.$$

Remark 13.3.

- T_ξ sends $E^{(k)}$ to $E^{(k+1)}$.
- T_ξ is adjointable:
$$T_\xi^*(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n) := \phi(\langle \xi, \eta_1 \rangle)\eta_2 \otimes \cdots \otimes \eta_n.$$
In particular, $T_\xi^*(a) = 0$ for all $a \in A$.

Definition 13.4. The *Toeplitz–Pimsner algebra* of (E, ϕ), denoted by $\mathcal{T}_E$, is the smallest C*-algebra of $\mathbb{B}_A(E_+)$ containing T_ξ for all $\xi \in E$.

Theorem 13.5. *The Toeplitz–Pimnser algebra $\mathcal{T}_E$ is universal in the following sense:*

If (E, ϕ) is full over A, i.e. $\overline{\langle E, E\rangle} = A$. Let C be a C**-algebra and $\psi\colon A \to C$ be a *-homomorphism. If there exists $t_\xi \in C$ for all $\xi \in E$, such that*

$$\alpha t_\xi + t_\eta = t_{\alpha\xi+\eta}, \quad t_\xi\psi(a) = t_{\xi a}, \quad \psi(a)t_\xi = t_{\psi(a)}\xi, \quad t_\xi^* t_\eta = \psi(\langle \xi, \eta \rangle),$$

for all $\alpha \in \mathbb{C}$, $\xi, \eta \in E$.

Then ψ factor through $\tilde{\psi}\colon \mathcal{T}_E \to C$ sending T_ξ to t_ξ.

13.2 Cuntz–Pimsner algebras

Lemma 13.6. *If* $\operatorname{im}\phi \subseteq \mathbb{K}_A(E_+)$*. Then* $\mathbb{K}_A(E_+) \subseteq \mathcal{T}_E$*.*

Definition 13.7. The Cuntz–Pimsner algebra of a C*-correspondence (E, ϕ) is the quotient $\mathcal{O}_E$ given by the Cuntz–Pimsner extension

$$\mathbb{K}_A(E_+) \rightarrowtail \mathcal{T}_E \twoheadrightarrow \mathcal{O}_E.$$

Example 13.8. Let $A = \mathbb{C}$, $E = \mathbb{C}$ and ϕ is the multiplication. Then $T := T_1$ is the unilateral shift on $\ell^2(\mathbb{N})$: $Te_n = e_{n+1}$. The Cuntz–Pimsner extension becomes the well-known Toeplitz extension:

$$\mathbb{K}(\ell^2(\mathbb{N})) \rightarrowtail \mathcal{T} \twoheadrightarrow \mathrm{C}(\mathbb{T}).$$

This extension is semi-split. The cpc-section is given by the Hardy projection $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$.

Example 13.9. Let $A = \mathbb{C}$, $E = \mathbb{C}^n$ and ϕ is the multiplication. Then $\mathcal{T}_E$ is the C*-algebra $\mathrm{C}^*(V_1, \ldots, V_n)$ generated by n isometries $V_1, \ldots, V_n$ satisfying $\sum_i V_i V_i^* \leq 1$ and $\sum_i V_i V_i^* \neq 1$. The Cuntz–Pimsner extension becomes

$$\mathbb{K} \rightarrowtail \mathrm{C}^*(V_1, \ldots, V_n) \twoheadrightarrow \mathcal{O}_n,$$

where $\mathcal{O}_n$ is the Cuntz algebra.

Example 13.10. More generally, let E be a finitely-generated projective module over A. Then there exists a frame $\{\eta_i\}_{i=1}^n$ such that

$$\phi(a)\eta_j = \sum_{i=1}^n \eta_i \langle \eta_i, \phi(a)\eta_j \rangle.$$

The Cuntz–Pimsner algebra $\mathcal{O}_E$ is the C*-algebra generated by A together with n operators $S_1, \ldots, S_n$ satisfying

$$S_i^* S_j = \langle \eta_i, \eta_j \rangle, \quad \sum_{i=1}^n S_i S_i^* = 1, \quad aS_j = \sum_{i=1}^n \langle \eta_i, \eta_j \rangle.$$

As a special case, for $\mathbb{C}^n$ with the standard orthonormal basis one recovers the Cuntz algebra $\mathcal{O}_n$.

Example 13.11. Let A be a C^*-algebra, viewed as a Hilbert A-module. Let $\alpha\colon A \to A$ be an automorphism. Then (A, α) is a C^*-correspondence over A. The Cuntz–Pimsner algebra $\mathcal{O}_E$ is the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{Z}$. The Toeplitz–Pimsner algebra $\mathcal{T}_E$ in this case is usually denoted by $\mathcal{T}(A, \alpha)$. We shall see that $\mathcal{T}(A, \alpha)$ is KK-equivalent to A.

Theorem 13.12. *The Cuntz–Pimsner algebra $\mathcal{O}_E$ is universal in the following sense:*

If (E, ϕ) is full over A, i.e. $\overline{\langle E, E\rangle} = A$. Let C be a C^*-algebra and $\psi\colon A \to C$ be a *-homomorphism. If there exists $t_\xi \in C$ for all $\xi \in E$, such that

$$\alpha t_\xi + t_\eta = t_{\alpha\xi+\eta}, \quad t_\xi\psi(a) = t_{\xi a}, \quad \psi(a)t_\xi = t_{\psi(a)}\xi, \quad t_\xi^* t_\eta = \psi(\langle \xi, \eta\rangle), \quad \psi^{(1)}\phi(a) = \psi(a),$$

for all $\alpha \in \mathbb{C}$, $\xi, \eta \in E$. Here $\phi^{(1)}$ is defined as the map

$$\psi^{(1)}\colon A \cong E \otimes E^* \to \mathbb{K}_A(E), \qquad \xi \otimes \eta^* \mapsto T_\xi T_\eta^*.$$

Then ψ factor through $\tilde{\psi}\colon \mathcal{O}_E \to C$ sending T_ξ to t_ξ.

13.3 Pimsner–Voiculescu exact sequence

Let

$$\mathbb{K}_A(E_+) \rightarrowtail \mathcal{T}_E \twoheadrightarrow \mathcal{O}_E$$

be a Cuntz–Pimsner extension. Additionally, we require that

- $\operatorname{im}\phi \subseteq \mathbb{K}_A(E)$.
- (E, ϕ) is full. This implies that (E_+, ϕ_+) is also full.

Then $\mathbb{K}_A(E_+)$ is Morita equivalent to A. The equivalence is implemented by E_+. Hence $[E_+]$ defines an element in $\mathrm{KK}(\mathbb{K}(E_+), A) = \mathrm{KK}(A, A)$.

Theorem 13.13 (Pimsner)**.** *The inclusion map $i\colon A \hookrightarrow \mathcal{T}_E$ gives a* KK*-equivalence $[i] \in \mathrm{KK}(A, \mathcal{T}_E)$.*

Theorem 13.14. *The following diagram commutes in the Kasparov category:*

$$\begin{array}{ccccc} \mathbb{K}(E_+) & \hookrightarrow & \mathcal{T}_E & \twoheadrightarrow & \mathcal{O}_E \\ \downarrow{\scriptstyle [E_+]} & & \downarrow{\scriptstyle [i]^{-1}} & & \| \\ A & \xrightarrow{1-[E]} & A & \hookrightarrow & \mathcal{O}_E. \end{array}$$

Corollary 13.15. *There is a long exact sequence in* K*-theory:*

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{1-E_*} & \mathrm{K}_0(A) & \xrightarrow{\iota_*} & \mathrm{K}_0(\mathcal{O}_E) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(\mathcal{O}_E) & \xleftarrow{\iota_*} & \mathrm{K}_1(A) & \xleftarrow{1-E_*} & \mathrm{K}_1(A), \end{array}$$

where ι_* is the map in K*-theory in induced by the inclusion $\iota\colon A \hookrightarrow \mathcal{O}_E$, and E_* is the map in* K*-theory induced by the C^*-correspondence (E, ϕ).*

References

[1] Francesca Arici and Bram Mesland. "Toeplitz extensions in noncommutative topology and mathematical physics". In: *Geometric methods in physics XXXVIII*. Trends Math. Birkhäuser/Springer, Cham, 2020, 3–29. doi: 10.1007/978-3-030-53305-2_1. url: https://doi.org/10.1007/978-3-030-53305-2_1 (p. 10)

[2] Saad Baaj and Pierre Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les C*-modules hilbertiens. *C. R. Acad. Sci. Paris Sér. I Math.*, **296**:21 (1983), 875–878. issn: 0249-6291 (p. 40)

[3] Bruce Blackadar. K*-theory for operator algebras*. Second. Vol. 5. Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, 1998, xx+300. isbn: 0-521-63532-2 (pp. 1, 34)

[4] Man Duen Choi and Edward G. Effros. The completely positive lifting problem for C*-algebras. *Ann. of Math. (2)*, **104**:3 (1976), 585–609. issn: 0003-486X. doi: 10.2307/1970968. url: https://doi.org/10.2307/1970968 (pp. 36, 48)

[5] A. Connes and G. Skandalis. The longitudinal index theorem for foliations. *Publ. Res. Inst. Math. Sci.*, **20**:6 (1984), 1139–1183. issn: 0034-5318. doi: 10.2977/prims/1195180375. url: https://doi.org/10.2977/prims/1195180375 (pp. 24, 36)

[6] Alain Connes and Nigel Higson. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, **311**:2 (1990), 101–106. issn: 0764-4442 (p. 51)

[7] Joachim Cuntz. Simple C*-algebras generated by isometries. *Comm. Math. Phys.*, **57**:2 (1977), 173–185. issn: 0010-3616. url: http://projecteuclid.org.ezproxy.leidenuniv.nl:2048/euclid.cmp/1103901288 (p. 73)

[8] Joachim Cuntz. "K-theory and C*-algebras". In: *Algebraic* K*-theory, number theory, geometry and analysis (Bielefeld, 1982)*. Vol. 1046. Lecture Notes in Math. Springer, Berlin, 1984, 55–79. doi: 10.1007/BFb0072018. url: https://doi.org/10.1007/BFb0072018 (pp. 8, 10)

[9] Joachim Cuntz. A new look at KK-theory. K*-Theory*, **1**:1 (1987), 31–51. issn: 0920-3036. doi: 10.1007/BF00533986. url: https://doi.org/10.1007/BF00533986 (p. 57)

[10] Joachim Cuntz and Wolfgang Krieger. A class of C*-algebras and topological Markov chains. *Invent. Math.*, **56**:3 (1980), 251–268. issn: 0020-9910. doi: 10.1007/BF01390048. url: https://doi.org/10.1007/BF01390048 (p. 73)

[11] Michael Frank and David R. Larson. Frames in Hilbert C*-modules and C*-algebras. *J. Operator Theory*, **48**:2 (2002), 273–314. issn: 0379-4024 (p. 18)

[12] Nigel Higson. A characterization of KK-theory. *Pacific J. Math.*, **126**:2 (1987), 253–276. issn: 0030-8730. url: http://projecteuclid.org.ezproxy.leidenuniv.nl:2048/euclid.pjm/1102699804 (p. 50)

[13] Nigel Higson and John Roe. *Analytic* K*-homology*. Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2000, xviii+405. isbn: 0-19-851176-0 (p. 5)

[14] Kjeld Knudsen Jensen and Klaus Thomsen. *Elements of* KK*-theory*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1991, viii+202. isbn: 0-8176-3496-7. doi: 10.1007/978-1-4612-0449-7. url: https://doi.org/10.1007/978-1-4612-0449-7 (pp. 1, 25)

[15] G. G. Kasparov. The operator K-functor and extensions of C*-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, **44**:3 (1980), 571–636, 719. issn: 0373-2436 (p. 24)

[16] Dan Kucerovsky. The KK-product of unbounded modules. K*-Theory*, **11**:1 (1997), 17–34. issn: 0920-3036. doi: 10.1023/A:1007751017966. url: https://doi.org/10.1023/A:1007751017966 (p. 41)

[17] E. C. Lance. *Hilbert* C**-modules*. Vol. 210. London Mathematical Society Lecture Note Series. A toolkit for operator algebraists. Cambridge University Press, Cambridge, 1995, x+130. isbn: 0-521-47910-X. doi: 10.1017/CBO9780511526206. url: https://doi.org/10.1017/CBO9780511526206 (pp. 13, 16, 17)

[18] Ralf Meyer. “Categorical aspects of bivariant K-theory”. In: K-*theory and noncommutative geometry*. EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2008, 1–39. doi: 10.4171/060-1/1. url: https://doi.org/10.4171/060-1/1 (p. 50)

[19] Ralf Meyer. Homological algebra in bivariant K-theory and other triangulated categories. II. *Tbil. Math. J.*, **1**: (2008), 165–210. issn: 1875-158X (p. 50)

[20] Ralf Meyer and Ryszard Nest. The Baum–Connes conjecture via localisation of categories. *Topology*, **45**:2 (2006), 209–259. issn: 0040-9383. doi: 10.1016/j.top.2005.07.001. url: https://doi.org/10.1016/j.top.2005.07.001 (pp. 50, 56, 57)

[21] Ralf Meyer and Ryszard Nest. “Homological algebra in bivariant K-theory and other triangulated categories. I”. In: *Triangulated categories*. Vol. 375. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2010, 236–289 (pp. 50, 55, 58, 59)

[22] Michael V. Pimsner. “A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by **Z**”. In: *Free probability theory (Waterloo, ON, 1995)*. Vol. 12. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 1997, 189–212 (p. 73)

[23] Iain Raeburn. *Graph algebras*. Vol. 103. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005, vi+113. isbn: 0-8218-3660-9. doi: 10.1090/cbms/103. url: https://doi.org/10.1090/cbms/103 (pp. 68, 73)

[24] N. E. Wegge-Olsen. K-*theory and* C^**-algebras*. Oxford Science Publications. A friendly approach. The Clarendon Press, Oxford University Press, New York, 1993, xii+370. isbn: 0-19-859694-4 (pp. 8, 31)