Noncommutative geometry of foliations

NCG-Leiden

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Contents

Part I

Introduction to foliations and their C*-algebras

In this part, we provide several equivalent definitions of a foliation of a manifold, and introduce the holonomy and the holonomy groupoid of a foliation. Then we define the C*-algebra of a foliation, which is the reduced groupoid C*-algebra of its holonomy groupoid. The subtlety here comes from the possible non-Hausdorffness of the holonomy groupoid. This differs from the usual set-up which is covered in last year's groupoid seminar [16].

We mainly use [5, 14] for the geometry side of foliations, and refer to [8, 15] for their C*-algebras. The groupoid approach to C*-algebras was first introduced by Renault [21], and the papers [12, 23] are also useful for our purpose.

List of talks

1. Foliations: motivations, definitions and examples (Yuezhao Li, 12/09/2023)

   We start with some motivations for studying foliations, especially their noncommutative geometry. Then we provide several equivalent definitions of foliations and basic examples arising from geometry and dynamical systems following mostly [14, Chapter 1].

2. Holonomy and stability (Yufan Ge, 19/09/2023)

   We study the concepts of holonomy and stability, discuss in great detail the foliation of the Möbius bundle and the Reed foliation. Several stability theorems are given as well. The references are [14, Chapter 2] and [5, Chapter IV].

3. Lie groupoids (Jack Ekenstam, 26/09/2023)

   We study Lie groupoids and their related constructions following [14, Chapter 5], in parallel with those similar statements with topological groupoids in the groupoid seminar [16]. We describe the holonomy groupoid and the monodromy groupoid constructed from a foliation, which will be later used in the C*-algebras of foliations.

4. C*-algebras of foliations (Bram Mesland, 17/10/2023)

   The C*-algebra of a foliation is the groupoid C*-algebra of the holonomy groupoid, which may be non-Hausdorff. Previously in the groupoid seminar we were focusing only on Hausdorff groupoid. In this talk we focus on non-Hausdorff (but still locally Hausdorff) groupoids, and describe the construction of their C*-algebras. The main references are [15, Chapter VI] and the papers [12, 23].

September 12, 2023

Foliations: motivations, definitions and examples

Speaker: Yuezhao Li (Leiden University)

1.1 Motivations

1.1.1 From Frobenius theorem to foliations

A *foliation* on a manifold is, roughly speaking, a decomposition of it into immersed submanifolds (called the *leaves* of the foliation), such that the leaves fit together nicely. The theory of foliations is a tremendous component in modern differential geometry, contributed by several big names: Ehresmann, Reeb, Haefliger, Novikov, Thurston, Molino, Sullivan, Connes and many others.

The idea of foliations comes from a much older task: solving differential equations. This can be best by the celebrated Frobenius theorem[1]. Let M be a smooth manifold. Let E be a smooth rank-k distribution over M, that is, a rank-k subbundle of TM. An *integral manifold* of E is an immersed submanifold $N \subseteq M$ such that $T_pN = E_p$ for every $p \in N$. A distribution E is said to be *integrable* if every $x \in M$ is contained in an integral manifold of E. A distribution E is said to be *involutive* if $\Gamma(E)$, the space of smooth sections of E, is a Lie subalgebra of $\Gamma(TM)$ under the Lie bracket.

Theorem 1.1 (Frobenius)**.** *A distribution is integrable iff it is involutive.*

The Frobenius theorem is a vast generalisation of the classical existence theorems in differential equations, that is, the existence (and uniqueness) of solution (integral curve) of linear partial differential equations. The involutivity of a distribution can be viewed as its "local integrability", which by Frobenius theorem implies the existence of an integral manifold at every point. In modern language, the Frobenius theorem says that an involutive distribution gives a foliation of the base manifold, whose leaves are those maximal connected integral submanifolds.

Geometry serves as a big source of foliations. Every submersion of manifolds define a foliation on the total space, whose leaves are the connected components of fibres (Example 1.7). In particular, a fibre bundle defines a foliation of the total space. Another interesting example is the symplectic foliation of a Poisson manifold, which is generated by its Hamiltonian vector fields.

Dynamical systems also supply many interesing instances of folitions. Let G be a Lie group that acts on a smooth manifold M. Suppose that the stabliser groups have constant dimension. Then M is foliated by the connected components of G-orbits (Example 1.8). The dimension condition is necessary here as otherwise the leaves will not have the same dimension. That would be an instance of a singular foliation and an orbifold.

1.1.2 Why noncommutative geometry?

Let $(M, \mathscr{F})$ be a foliated manifold, that is, a manifold together with a foliation thereon. We wish to study the geometry of two "spaces":

1. The geometry of the leaves (generic fibre) L.
2. The geometry of the space of leaves $M/\mathscr{F}$, obtained from the equivalence relation generated by leaves.

However, there is several technical difficulty for us to study them, even in many simplest examples.

- A leaf L of a foliation can be usually non-compact. For example, every leaf of the Kronecker foliation (Example 1.17) is diffeomorphic to $\mathbb{R}$. This non-compactness, in many cases, obstructs us to pass from local (leaves) to global (the space M).

[1] Despite of the name, this theorem is indeed not proven by Frobenius, at whose age the concept of manifolds were not yet established.

- The quotient space $M/\mathscr{F}$ is usually badly behaved as a topological space. In the case of the Kronecker foliation, every leaf is dense in $\mathbb{T}^2$. So the quotient space is necessarily non-Hausdorff.
- We would like to define a measure on the quotient space, so that we can (1) talk about integrals (2) make the word "generic" precise, in the sense of probability. But $M/\mathscr{F}$ is also quite terrible as a measurable space due to ergodicity. That is, every measurable function of $M/\mathscr{F}$ is necessarily constant almost everywhere. Hence $L^p(M/\mathscr{F}) = \mathbb{C}$ for every $p \in [1, \infty]$.

These force us to seek other geometric objects to replace $M/\mathscr{F}$, and leads to Connes' C^*-algebras of foliations and noncommutative geometry. Roughly, these can be summarised as follows:

- The leaves of a foliation generate an equivalence relation of the manifold, and hence a *groupoid* $\mathrm{Hol}(M, \mathscr{F})$, called the holonomy groupoid of the foliation. This is a noncommutative object which encodes the topology of a foliation.
- By some standard techniques introduced by Renault, one can define a C^*-algebra $\mathrm{C}^*(M, \mathscr{F})$ of this groupoid. This is known as Connes' C^*-algebra of foliation.
- A measure on $M/\mathscr{F}$ should be by a measure on M that is holonomy invariant. This is called a transverse measure of the foliation. With such as input, one may get a trace on $\mathrm{C}^*(M, \mathscr{F})$ and hence an "index map" $\mathrm{K}_*(\mathrm{C}^*(M, \mathscr{F})) \to \mathbb{R}$. This reveals the index theory of a foliation.

Thus we have the following nice machine for foliations:

Foliated manifold $(M, \mathscr{F})$	$\Longrightarrow$	Holonomy groupoid $\mathrm{Hol}(M, \mathscr{F})$	$\Longrightarrow$	Foliation C^*-algebra $\mathrm{C}^*(M, \mathscr{F})$

However, a new question emerges: what are the correct *morphisms*? Note that assigning the groupoid C^*-algebra to a groupoid is not functorial. For example, topological spaces and topological groups are both topological groupoids. Whereas groupoid C^*-algebras for spaces are contravariantly functorial, those for groupois are covariantly functorial. So homomorphisms of groupoids are not satisfying, and we will replace them by a more general concepts of morphisms: the *groupoid correspondences*. These were first introduced in [11] and later referred to as "Hilsum–Skandalis morphisms". A brief account of this is Bram's talk at the groupoid seminar [16].

1.2 Definitions and first examples

1.2.1 By foliation atlas

Definition 1.2 (First definition)**.** Let M be a smooth manifold of dimension n. A (regular) *foliation atltas* of codimension q is an atlas of M, each chart of the form

$$(U_i,\ \varphi_i : U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q),$$

such that the transition functions

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

have the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)), \qquad x \in \mathbb{R}^{n-q},\ y \in \mathbb{R}^q$$

for some smooth functions $g_{ij} : \mathbb{R}^n \to \mathbb{R}^{n-q}$ and $h_{ij} : \mathbb{R}^q \to \mathbb{R}^q$.

A (regular) *foliation* of codimension q of M is a maximal foliation atlas $\mathscr{F}$ of M. We write $\operatorname{codim} \mathscr{F}$ for the codimension of $\mathscr{F}$.

A *foliated manifold* is a pair $(M, \mathscr{F})$ of a manifold M together with a foliation $\mathscr{F}$ of it.

Definition 1.3. A *plaque* of a foliation chart $(U_i,\ \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ is a connected component of the submanifold $\varphi^{-1}(\mathbb{R}^{n-q} \times \{y\})$ for some $y \in \mathbb{R}^q$. Two points $x, y \in M$ belong to the same *leaf* if there exist a sequence of foliation charts $U_1, \ldots, U_k$ and a sequence of points $x = p_1, \ldots, p_k = y$ with $p_i \in U_i$ and such that p_{i-1} and p_i belong to the same plaque.

A *leaf* is a collection of points that belong to the same leaf.

The *space of leaves* is the quotient space $M\mathscr{F}$ obtained by the equivalence relation generated by the leaves.

One can easily show that leaves are immersed submanifolds of M:

Proposition 1.4. *Let $(M, \mathscr{F})$ be a codimension-q foliated manifold. Then every leaf is an immersed submanifold of M of dimension $n-q$.*

Proof. Choose $x \in M$ and let L be the leaf of x. Every plaque is a smooth chart of dimension $n-q$. If x belongs to a plaque, then every point of the plaque belongs to L by definition. This holds for other plaques containing a point that belongs to the same leaf with x. Therefore L is covered by these plaques and they are smoothly compatible. The plaques are immersed submanifolds of M, and this holds for L as well. □

Definition 1.5. Let $(M, \mathscr{F})$ and $(N, \mathscr{G})$ be foliated manifolds. A morphism between them is a smooth map $f : M \to N$ which preserves the foliation. That means, every leaf of $\mathscr{F}$ is mapped into a leaf of N. We also say such a map is foliated.

Example 1.6 (Trivial foliation). The space $\mathbb{R}^n$ admits a trivial foliation: the foliated atlas consists of a single chart $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^{n-q} \times \mathbb{R}^q$. Similarly, any linear isomorphism $A\colon \mathbb{R}^n \to \mathbb{R}^n$ determines a foliation. The leaves are given by $A^{-1}(\mathbb{R}^{n-q} \times \{y\})$.

Example 1.7 (Submersions). Let $\pi\colon E \to M$ be a submersion. This defines a foliation of E whose leaves are connected components of the fibres of π. The codimension of the foliation equals $\dim M$. If every fibre is connected, then $E/\mathscr{F} = M$. If some fibres are not connected, then $E/\mathscr{F}$ is a quotient of M which is necessarily non-Hausdorff.

In particular, a rank-k vector bundle $\pi\colon E \to M$ gives a foliation of E whose leaves are given by $\pi^{-1}(x) \simeq \mathbb{R}^k$.

Example 1.8 (Lie group actions). Another important source of interesting foliations is smooth dynamical systems. Let G be a Lie group which acts on a smooth manifold M. We wish to foliate M into the connected components of the orbits under the G-action. But this cannot be always true. For example, consider the canonical action of $\mathrm{GL}(2, \mathbb{R})$, or $\mathrm{SL}(2, \mathbb{R})$, or $\mathbb{C}^*$ (viewed as a submanifold of $\mathbb{R}^2$ equipped with its usual product) on $\mathbb{R}^2$. The orbits are $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. However, if it were a foliation, then these orbits should have the same dimension.

In order to exclude such singular cases, consider the isotropy subgroups at x:

$$G_x := \{g \in G \mid gx = x\}$$

This is a closed subgroup, hence a Lie subgroup. The orbit of x is identified with the homogeneous space G/G_x, and immersed into M. We say that the action of G on M is *foliated*, if G_x has constant dimension. Then all orbits have the same dimension, and the orbits form a foliation of M.

For example, let $G = \mathbb{R}$. Then an $\mathbb{R}$-action on M

$$\mu\colon \mathbb{R} \times M \to M$$

is also called a *flow*. The vector field associated to the flow is

$$X(x) := \left.\frac{\partial \mu(t, x)}{\partial t}\right|_{t=0}.$$

An $\mathbb{R}$-action is foliated, iff its associated vector field X is nowhere vanishing. Then the leaves of the foliation are given by the integral curves of X.

1.2.2 By Haefliger structure

A foliation can be equivalently described in several equivalent ways. The following definition is based on a "Haefliger structure".

Definition 1.9 (Second definition)**.** A codimension-q foliation of a manifold M is given by an open cover $\{U_i\}$ of M together with submersions

$$s_i\colon U_i \to \mathbb{R}^q,$$

such that there are (necessarily unique) diffeomorphisms

$$\gamma_{ij}\colon s_j(U_i \cap U_j) \to s_i(U_i \cap U_j)$$

satisfying:

$$\gamma_{ij} \circ s_j = s_i, \quad \gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}.$$

These maps γ_{ij} are called the *Haefliger cocycles* of the foliation. We briefly call the data $\{U_i, s_i, \gamma_{ij}\}_{ij}$ a *Haefliger structure*.

Equivalence with first definition. Suppose we are given a Haefliger structure $\{U_i, s_i, \gamma_{ij}\}_{ij}$. Let $\{(V_k, \varphi_k)\}_k$ be an atlas of M. Up to refinement, we may assume that each V_k is contained in a single U_{i_k}. Then $s_{i_k}|_{V_k}$ is also a submersion, hence there is a diffeomorphism $\kappa_k\colon s_{i_k}(V_k) \to \mathbb{R}^q$ such that $\kappa_k \circ s_{i_k}$ retricts to the projection $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ on local charts. That is, $\kappa_k \circ s_{i_k} \circ \varphi_k^{-1} = \mathrm{pr}_{\mathbb{R}^q}$. We claim that $\{V_k, \varphi_k\}$ is a foliation chart in the sense of Definition 1.2: we have

$$\begin{aligned}
\mathrm{pr}_{\mathbb{R}^q} \circ \varphi_{kl}(x,y) &= \mathrm{pr}_{\mathbb{R}^q} \circ \varphi_k \circ \varphi_l^{-1}(x,y) \\
&= \kappa_k \circ s_{i_k} \circ \varphi_l^{-1}(x,y) \\
&= \kappa_k \circ \gamma_{i_k i_l} \circ s_{i_l} \circ \varphi_l^{-1}(x,y) \\
&= \kappa_k \circ \gamma_{i_k i_l} \circ \kappa_l^{-1}(y).
\end{aligned}$$

Conversely, given a foliation atlas $\{U_i, \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q\}$ such that $\varphi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y))$. Set $s_i := \mathrm{pr}_{\mathbb{R}^q} \circ \varphi_i$ and $\gamma_{ij} := h_{ij}$. These render the desired Haefliger cocycles. □

Example 1.10. Let $\pi\colon E \to M$ be a rank-$(n-q)$ vector bundle and $\dim M = q$. This gives a codimension-q foliation on M whose leaves are $\pi^{-1}(x)$ for $x \in M$. Let $\{U_i, \varphi_i\colon U_i \to \mathbb{R}^q\}$ be an atlas of M, and $\{\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^{n-q}\}$ be the associated local trivialisations of E. Then the foliation chart on $\pi^{-1}(U_i)$ is just the composition of the local trivialisations Φ_i, with the chart of U_i. The Haefliger cocycles are just the transition functions $h_{ij}\colon U_i \cap U_j \to \mathrm{GL}(q, \mathbb{R})$ of the base manifold M.

1.2.3 By involutive distribution

The alternative definition below justifies the connection between foliations and the Frobenius theorem:

Definition 1.11 (Third definition)**.** A codimension-q foliation $\mathscr{F}$ of a manifold M is given by an involutive distribution.

The involutive distribution is called the *tangent bundle* of the foliation $\mathscr{F}$ and denoted by $T\mathscr{F}$.

The equivalence with the first definition is heavily based on the Frobenius theorem. We will omit the proof and only sketch how to derive these definitions from each other. The proof can be found in [14, Section 1.2]; see also [13, Theorem 19.21].

Let $E \subseteq TM$ be an involutive distribution. By Frobenius theorem, it is integrable: every point $x \in M$ belongs to an integral manifold. These manifolds form a foliation of M.

Conversely, if $(M, \mathscr{F})$ is a foliated manifold. For every $x \in M$, let L be the leaf of x. Then the collection $\{T_x L\}_{x \in M}$ is a vector subbundle of TM which is involutive. We denote this bundle by $T_x\mathscr{F}$ and also write $T_x\mathscr{F}$ for $T_x L$.

Example 1.12. A vector bundle $E \to M$ gives a foliation $\mathscr{F}$ on the total space E. The tangent bundle of the foliation $T\mathscr{F}$ is just the vertical bundle of $E \to M$: $VE := \ker T\pi$. Note that every leaf is vertical in the sense that $T\mathscr{F} \subseteq \ker T\pi$.

Example 1.13. Let G be a Lie group which has a foliated action on M. Then $T\mathscr{F}$ is spanned by the fundamental vector fields of this group action. Namely, the image of the bundle homomorphism

$$\mathfrak{g} \times M \to TM, \qquad (\xi, x) \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x\exp(t\xi).$$

Remark 1.14. We have seen so far that a Lie group action generates a foliation necessarily when the dimensions of the stabliser groups are constant. If this is not the case, then we speak of a *singular foliation*. The following definition (due to Peter Stefan, Iakovos Androulidakis and Georges Skandalis) is modelled on the third definition that we have introduced above.

Definition 1.15. A *(singular) foliation* is a locally finitely generated, involutive $\mathrm{C}^\infty(M)$-submodule of $\Gamma_c(M, TM)$.

Singular foliations are much more complicated to deal with as opposed to those regular ones. For example, the holonomy groupoid of a singular foliation is hard to define. It seems that the most general definition so far is the one given in [1]. We might talk about this in a later talk.

1.3 Constructions and more examples

1.3.1 Product foliations

Let $(M, \mathscr{F})$ and $(N, \mathscr{G})$ be foliated manifolds, given by a Haefliger structure $(U_i, s_i, \gamma_{ij})_{ij}$ on M and $(U'_k, s'_k, \gamma'_{kl})$ on N. There is a *product foliation* $\mathscr{F} \times \mathscr{G}$ on $M \times N$, given by the Haefliger structure

$$(U_i \times U'_k, s_i \times s'_k, \gamma_{ij} \times \gamma'_{kl})_{ijkl}.$$

The product foliation $\mathscr{F} \times \mathscr{G}$ has codimension $\operatorname{codim}\mathscr{F} + \operatorname{codim}\mathscr{G}$. The tangent bundle $T(\mathscr{F} \times \mathscr{G}) \simeq T\mathscr{F} \oplus T\mathscr{G} \subseteq TM \oplus TN \simeq T(M \times N)$.

1.3.2 Pullback foliations along transverse maps

Definition 1.16. Let $(M, \mathscr{F})$ be a foliated manifold. A smooth map $f : N \to M$ is *transverse* to $\mathscr{F}$, if f is transverse to all the leaves of $\mathscr{F}$. That is, for every $x \in N$,

$$T_{f(x)}M = T_{f(x)}\mathscr{F} + T_x f(T_x N).$$

Let $\mathscr{F}$ be given by the Haefliger structure $(U_i, s_i, \gamma_{ij})_{ij}$. We claim that the data

$$\big(V_i := f^{-1}(U_i), \quad s'_i := s_i \circ f|_{V_i}, \quad \gamma_{ij}\big)_{ij}$$

defines a Haefliger structure. Its determined foliation is called the *pullback foliation* on N along f, denoted by $f^*\mathscr{F}$.

Proof. We must show that the maps $s'_i \colon V_i \to \mathbb{R}^q$ are submersion, that is, $T_x s'_i = T_{f(x)}s_i \circ T_x f$ is surjective for every $x \in V_i$. Since f is transverse to $\mathscr{F}$, we have that the map

$$\tilde{f} : T_x V_i \xrightarrow{T_x f} T_{f(x)}U_i \to T_{f(x)}U_i / T_{f(x)}\mathscr{F}$$

is surjective. The submersion s_i is trivial along the leaves, that is, $T_{f(x)}s_i = 0$ on $T_{f(x)}\mathscr{F} \subseteq T_{f(x)}U_i$. So $T_x s_i$ factors through the quotient map $T_{f(x)}U_i \to T_{f(x)}U_i / T_{f(x)}\mathscr{F}$. Thus the diagram

$$\begin{array}{ccccc} T_x V_i & \xrightarrow{T_x f} & T_{f(x)}U_i & \xrightarrow{T_{f(x)}s_i} & \mathbb{R}^q \\ & \searrow^{\tilde{f}} & \downarrow & \nearrow_{\tilde{s}_i} & \\ & & T_{f(x)}U_i / T_{f(x)}\mathscr{F} & & \end{array}$$

commutes. So $T_{f(x)}s_i \circ T_x f = \tilde{s}_i \circ \tilde{f}$ is the composition of two surjective maps, hence surjective. □

We have $\operatorname{codim} f^*\mathscr{F} = \operatorname{codim}\mathscr{F}$ and $Tf^*\mathscr{F} = (Tf)^{-1}(T\mathscr{F})$.

1.3.3 Quotient foliations of covering space actions

Let $(M, \mathscr{F})$ be a foliated manifold such that M carries a free, properly discontinuous action of a discrete Lie group G. Then the quotient manifold M/G is Hausdorff. Assume that $\mathscr{F}$ is *invariant* under the G-action, that is, every $g \in G$ is an automorphism of the foliation $(M, \mathscr{F})$. Then there is an induced foliation $\mathscr{F}/G$ on M/G as follows.

Since G acts freely and properly discontinuously, the quotient map $\pi\colon M \to M/G$ defines a covering space. Let $\{(U_i, \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$ be a foliation atlas of $\mathscr{F}$. Then π is a local diffeomorphism. We may assume that $\pi|_{U_i}$ is a diffeomorphism by replacing U_i by its refinement. Then it has an inverse section s_i and

$$\{(\varphi_i \circ s_i\colon \pi(U_i) \to \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$$

renders a foliation $\mathscr{F}/G$ for M/G. The leaves of $\mathscr{F}/G$ are quotient manifolds L/G_L, where L is a leaf of $\mathscr{F}$ and G_L is the isotropy subgroup of the leaf L. The latter is well-defined because every $g \in G$ maps a leaf into a leaf.

We have $\operatorname{codim}(\mathscr{F}/G) = \operatorname{codim}(\mathscr{F})$ and $T(\mathscr{F}/G) = T\pi(T\mathscr{F})$.

Example 1.17 (Kronecker foliation). Let $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$ and define the submersion

$$s\colon \mathbb{R}^2 \to \mathbb{R}, \qquad s(x, y) = x - \vartheta y.$$

Then it defines a foliation on $\mathbb{R}^2$ following Example 1.6. The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ freely and properly discontinuously, and the aforementioned foliation on $\mathbb{R}^2$ is $\mathbb{Z}^2$-invariant. So there is an induced foliation $\mathscr{F}$ on the quotient manifold $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. This is known as the *Kronecker foliation* on $\mathbb{T}^2$. Equivalently, one can consider the integral curves generated by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \vartheta$.

Example 1.18 (Foliation of the Möbius band). Recall that the Möbius bundle is the quotient space of $\mathbb{R}^2$:

$$\text{Möb} := \mathbb{R}^2/\sim, \qquad (x, y) \sim (x+1, -y).$$

The projection $\mathbb{R}^2 \to \text{Möb}$ is a two-fold covering space. The trivial codimension-1 foliation of $\mathbb{R}^2$ is invariant under this group action, hence there is a quotient foliation of Möb. The interesting property of this foliation is that every leaf (which is diffeomorphic to $\mathbb{T}$) will wrap around the whole band twice except for the middle one generated by the plaques with $y = 0$.

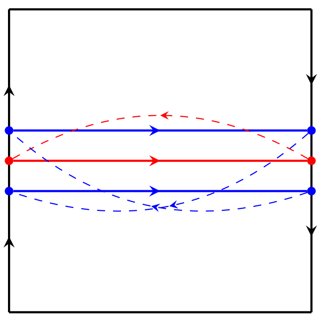


Figure 1.1: A foliation of the Möbius band

1.3.4 Suspensions

Suspensions of diffeomorphism groups are special cases of the quotient foliation. They consititute a large class of interesting foliations.

Let F be a smooth manifold and $f : F \to F$ be an automorphism. Consider the trivial 1-dimensional foliation on $M = \mathbb{R} \times F$, i.e. the leaves are $\mathbb{R} \times \{x\}$ for $x \in F$. There is a $\mathbb{Z}$-action on M generated by the diffeomorphism

$$(t, x) \mapsto (t + 1, f(x)).$$

The action is free and proper, and leaves the foliation on M invariant. So there is a quotient foliation on the quotient manifold $\mathbb{R} \times_{\mathbb{Z}} F$. This is called the *suspension* of the diffeomorphism f, or the suspension of the group $\mathbb{Z}$ generated by f.

1.3.5 Flat bundles

The suspensions are a special case of flat bundles. Let M be a manifold and $\widetilde{M}$ be the universal cover of M. Then $\widetilde{M} \to M$ is a (right) principal $\pi_1(M)$-bundle. Assume that $\pi_1(M)$ acts on another manifold F. Then there is an associated fibre bundle

$$\widetilde{M} \times_{\pi_1(M)} F \to M.$$

The space $\widetilde{M} \times_{\pi_1(M)} F$ is the quotient of $\widetilde{M} \times F$ under the equivalence relation $(x, y) \sim (xg^{-1}, gy)$.

The submersion $\widetilde{M} \times F \to F$ generates a foliation of $\widetilde{M} \times F$ which is invariant under the action of $\pi_1(M)$. So there is a quotient foliation on $\widetilde{M} \times_{\pi_1(M)} F$. This is called a *flat bundle*. In contrast to the foliation coming from a submersion, in which every leaf is vertical in the sense that $T\mathscr{F} \subseteq \ker T\pi$, in a flat bundle all leaves are horizontal, that is, $T_x\mathscr{F} \simeq T_{\pi(x)}M$.

The suspension is the special case where $M = \mathbb{T}$ and the action of $\pi_1(M) = \mathbb{Z}$ on F is generated by the diffeomorphism f.

1.3.6 Reeb foliation

Foliations can be defined on manifolds with boundary as well. But this requires that the foliation behaves nicely near the boundary, that is, the foliation is either *transverse* to the boundary or *tangent* to the boundary.

Definition 1.19 ([6, Definition 1.1.11]). Let $(M, \mathscr{F})$ be a foliated manifold, and $N \subseteq M$ be a submanifold. We say that:

- $\mathscr{F}$ is *transverse* to N, denoted by $\mathscr{F} \pitchfork N$, if every leaf of $\mathscr{F}$ is transverse to N. That is,

  $$T_xM = T_x\mathscr{F} + T_xN$$

  for every $x \in N$.
- $\mathscr{F}$ is *tangent* to N, if $T_x\mathscr{F} \subseteq T_xN$ for every $x \in N$. Equivalently this means every leaf L of $\mathscr{F}$ is either disjoint from N, or contained in N.

The *Reeb foliation* of the solid torus $X = \mathbb{D}^2 \times \mathbb{T}$ is a foliation on a manifold with boundary, of the second sort. Let

$$\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

be the unit disk. Foliate the solid cylinder $\mathbb{D}^2 \times \mathbb{R}$ by the submersion

$$f : \operatorname{Int}(\mathbb{D}^2) \times \mathbb{R} \to \mathbb{R}, \qquad f(x, y, t) = \exp\left(\frac{1}{1 - x^2 - y^2}\right) - t$$

and require that the boundary of the solid cylinder is another leaf. This gives a foliation of the solid cylinder.

The translation action of $\mathbb{Z}$ on the second entry of $\mathbb{D}^2 \times \mathbb{R}$ preserves this foliation. So there is a quotient foliation on solid torus $X = \mathbb{D}^2 \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{D}^2 \times \mathbb{T}$. This is the *Reeb foliation* on the solid torus. Note that every leaf is diffeomorphic to $\mathbb{R}^2$ except for the boundary leaf $\mathbb{T}^2$.

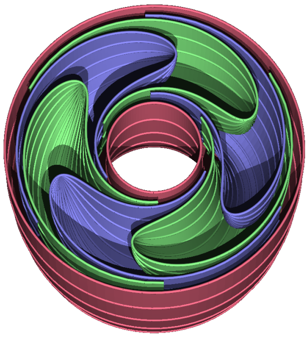


Figure 1.2: A section of Reeb foliation of the solid torus. Picture by Ilya Voyager from WIKIPEDIA (licensed under CC0 1.0 Deed).

Now we describe the Reeb foliation on $\mathbb{S}^3$. A statement from topology says that $\mathbb{S}^3$ can be obtained by gluing together two copies of the solid torus along the boundary $\mathbb{T}^2$:

$$\mathbb{S}^3 = X \cup_{\partial X} X, \qquad X = \mathbb{D}^2 \times \mathbb{T}, \qquad \partial X = \mathbb{T}^2.$$

The Reeb foliation can be glued together to form a foliation on $\mathbb{S}^3$ as well. This does happen in general: one needs certain "tameness" conditions in order to glue two foliations together along the boundary; we shall, however, not talk about this in detail. Then we obtain the *Reeb foliation* on $\mathbb{S}^3$. It has all leaves diffeomorphic to $\mathbb{R}^2$ except for the "boundary" leaf of solid cylinder, which is compact.

The importance of the Reeb foliation is illustrated by the following celebrated theorem:

Theorem 1.20 (Novikov's compact leaf theorem). *Every codimension-1 foliation of $\mathbb{S}^3$ has a compact leaf, bounding a solid torus with the Reeb foliation.*

So the Reeb foliation plays an important role in the foliation theory of $\mathbb{S}^3$.

September 19, 2023

Holonomy and stability

Speaker: Yufan Ge (Leiden University)

2.1 Holonomy

2.1.1 Motivation: Poincaré return map

The idea of holonomy goes back to the Poincaré return map. Let $(\phi, M, \mathbb{R})$ be a global dynamical system, where

$$\phi : \mathbb{R} \times M \to M$$

is the flow. Suppose that γ is a periodic orbit, that is, there exists a time T such that $\phi(T, x) = x$ for every $x \in \gamma$. The orbit is thus a circle. One is then motivated to study the property of this closed orbit. The idea of Poincaré is to look at how the points near this closed orbit behave with time. This is done through the Poincaré section and the Poincaré return map.

Let $p \in \gamma$ and let S be a local smooth section through p which is transverse to the flows of ϕ. Such an S is called a *Poincaré section*. A function $P : U \to S$ from an open and connected neighbourhood $U \subseteq S$ of p to S is called a *Poincaré map*. if the followings hold:

- $P(p) = p$.
- $P(U)$ is diffeomorphism to U.

- For every $x \in U$, $P(x) = \phi(t_0, x)$, where $t_0 = \min\{t \in \mathbb{R} \mid \phi(t,x) \in S\}$. Namely, t_0 is the first time that x returns to S.

The idea of holonomy is reminiscent of that of the Poincaré return maps, which describes the behaviour of all the nearby leaves of a given leaf.

2.1.2 The holonomy map over a path

Let $(M, \mathscr{F})$ be a foliated manifold. Let α be a smooth path in a leaf L of $\mathscr{F}$ from x to y. That is,

$$\alpha \colon [0,1] \to L, \qquad \alpha(0) = x, \quad \alpha(1) = y.$$

We wish to define the holonomy map as a local map "along" the leaves, "over" the path α, which takes x to y. More precisely, it is defined as an equivalence class of smooth maps called a germ:

Definition 2.1. Let M and N be smooth manifolds. Let $x \in M$ and $y \in N$. A *germ of smooth maps* from x to y is an equivalence class of smooth maps

$$f : U \to V, \quad \text{satisfying } f(x) = y$$

where U is an open neighbourhood of x and y is an open neighbourhood of y. Two smooth maps $f : U \to V$ and $f' \colon U' \to V'$ are equivalent as germs if there exists an open neighbourhood $W \subseteq U \cap U'$ of x such that $f|_W = f'|_W$. We write germ(f) for the germ of f.

Similarly, one can define a *germ of local diffeomorphisms* from x to y, by requiring in addition that f is a local diffeomorphism, that is, a diffeomorphism from an open neighbourhood of x to its image. Clearly, if f is a local diffeomorphism, then the germ (of diffeomorphisms) of f is the same thing as the germ of smooth maps of f.

Let $x \in M$. Denote the set of germs of local diffeomorphisms $f : (M, x) \to (M, x)$ by $\mathrm{Diff}_x(M)$. Clearly this is a group under the composition of smooth maps.

The holonomy of a leaf L along a path $\alpha \colon [0,1] \to L$ will be defined to be a germ of smooth maps from $x = \alpha(0)$ to $y = \alpha(1)$. In order to do that, a choice of transverse section across x, resp. y, is needed.

Definition 2.2. Let $(M, \mathscr{F})$ be a foliated manifold of codimension q. A *transverse section*, or a *transversal*, is a q-dimensional submanifold of M which is transverse to $\mathscr{F}$ (Definition 1.19).

Definition and Lemma 2.3. Let L be a leaf of $(M, \mathscr{F})$ and $\alpha \colon [0,1] \to L$ be a smooth path in L. Let $x = \alpha(0)$ and $y = \alpha(1)$. Let T and S be transverse sections of $(M, \mathscr{F})$ such that $x \in T$ and $y \in S$.

The *holonomy map* $\mathrm{hol}^{S,T}(\alpha)$ of the leaf L along the path α, with respect to the transversals T and S, is defined as follows.

1. Assume first that $\alpha([0,1])$ is contained in a single foliation chart. Then x and y belong to the same plaque. *There exists a smooth map $f : A \to S$, where $A \subseteq U$ is an open subset containing x, such that:*

   - *$f(x) = y$;*
   - *For any $x' \in A$, $f(x')$ and x' belongs to the same plaque.*
   - *f is a local diffeomorphism at x.*

   In particular, the germ of f is uniquely determined. Then we define the holonomy map as the germ of f:
   $$\mathrm{hol}^{S,T}(\alpha) := \mathrm{germ}(f).$$

2. In the general case, *there exists a finite sequence of foliation charts* $\{U_0, \dots, U_k\}$ *such that* $\alpha([\frac{i}{k}, \frac{i+1}{k}]) \subseteq U_i$. Set $x_i = \alpha(\frac{i}{k})$ and α_i to be the restriction of α to $[\frac{i}{k}, \frac{i+1}{k}]$. Choose transversals $T = T_0, T_1, \dots, T_n = S$ of $\mathscr{F}$ at $x = x_0, x_1, \dots, x_n = y$. Define

$$\mathrm{hol}^{S,T}(\alpha) := \mathrm{hol}^{T_k,T_{k-1}}(\alpha_{k-1}) \circ \mathrm{hol}^{T_{k-1},T_{k-2}}(\alpha_{k-1}) \circ \cdots \circ \mathrm{hol}^{T_1,T_0}(\alpha_0).$$

In particular, $\mathrm{hol}^{S,T}(\alpha)$ *does not depend on the intermediate transversals* $T_1, \dots, T_{k-1}$.

Detailed proofs of the stataments above can be found in [5, Chapter IV], which we omit here. To understand this construction, let us work in a foliation chart $(U, \varphi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ centered at x. Then x belongs to the plaque $\varphi^{-1}(\mathbb{R}^{n-q} \times \{0\})$. A transverse section S at x has dimension q and intersects $T_x\mathscr{F}$ transversely, which means that

$$T_xM = T_xS \oplus T_x\mathscr{F}.$$

Therefore, locally S intersects the leaf of x, and *every leaf* in a neighbourhood of x, at a single point. We may assume that U is small enough so that S intersects every leaf in U at a unique point. Then S is given by $\varphi^{-1}(\{0\} \times \mathbb{R}^q)$ up to a coordinate change.

Now assume that $\alpha\colon [0,1] \to L$ is contained in a single chart U. Assume without loss of generality that the transverse sections T at $x = \alpha(0)$ and S at $y = \alpha(1)$ intersect the every leaf in U at a unique point. Then the holonomy map $\mathrm{hol}^{S,T}(\alpha)$ is defined as follows: for any $x' \in T$, which belongs to a leaf L'. Then $\mathrm{hol}^{S,T}(\alpha)(x')$ is the unique point that belongs to the intersection $S \cap L'$.

These process can be done for paths which do not lie in a unique foliation chart as well, whereas the *holonomy groups* become significant as they might be non-trivial.

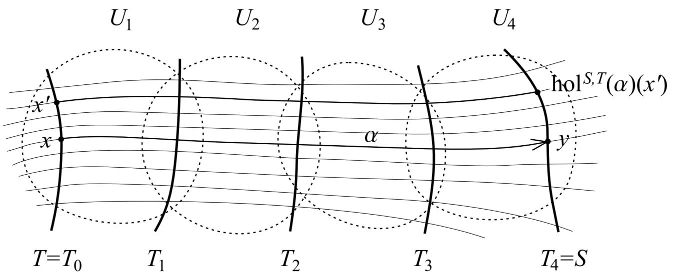


Figure 2.1: Holonomy maps. Picture from [14, Fig 2.1].

Proposition 2.4. *The holonomy maps have the following properties:*

1. *If* α *is a path in* L *from* x *to* y*, and* β *is a path in* L *from* y *to* z*. Let* T, S, R *be transverse sections of* $\mathscr{F}$ *at* x, y, z*. Then*
$$\mathrm{hol}^{R,T}(\beta\alpha) = \mathrm{hol}^{R,S}(\beta) \circ \mathrm{hol}^{S,T}(\alpha).$$

2. Homotopy invariance. *If* α *and* α' *are paths which are homotopic in a leaf with basepoints fixed. Then* $\mathrm{hol}^{S,T}(\alpha) = \mathrm{hol}^{S,T}(\alpha')$.

3. *Let* α *be a path in* L *from* x *to* y*. Let* T, T' *be transverse sections at* x *and* S, S' *be transverse sections at* y*. Then*
$$\mathrm{hol}^{S',T'}(\alpha) = \mathrm{hol}^{S',S}(\bar{y}) \circ \mathrm{hol}^{S,T}(\alpha) \circ \mathrm{hol}^{T,T'}(\bar{x}),$$
where $\bar{x}$ *(resp.* $\bar{y}$*) stands for the identity path at* x *(resp. at* y*).*

2.1.3 The honolomy group

Definition 2.5. Let $(M, \mathscr{F})$ be a foliated manifold and L a leaf of $\mathscr{F}$. The *holonomy group* of L at x with respect to the transverse section T, denoted by $\mathrm{Hol}(L, T, x)$, is defined as the image of the map

$$\mathrm{hol}^{T,T}\colon \pi_1(L, x) \to \mathrm{Diff}_x(T) \simeq \mathrm{Diff}_0(\mathbb{R}^q).$$

It is clear (indeed, from 3 of the previous proposition) that a different choice of T, or a different choice of $x \in L$, results in a conjugacy of the holonomy map by a germ of diffeomorphism. Therefore we may define $\mathrm{Hol}(L)$ to be the conjugacy class of the holonomy maps in all of those $\mathrm{Hol}(L, T, x)$ with $x \in L$ and T a transverse section at x.

The following result guarantees that it is sufficient to consider only $\mathrm{Im}(\mathrm{hol}^{T,T})$:

Theorem 2.6 (Transverse uniformity of leaves, [5, §III.2, Theorem 3]). *Let L be a leaf of $\mathscr{F}$ and $x, y \in L$. Then there exists transversals T at x and S at y, together with a local diffeomorphism f from an open subset $U \subseteq T$ to S, such that for any leaf L':*

$$f(L' \cap T) = L' \cap S.$$

Therefore, we only need to consider the first return map.

Example 2.7. If L is simply connected. Then $\pi_1(L, x) = 0$ and hence $\mathrm{Hol}(L) = 0$.

Example 2.8 (Foliation of the Möbius band). A foliation of the Möbius band, as in Example 1.18, has two distinct classes of leaves: the middle leaf "wraps around" the base only once, whereas every other leaf wraps around the base twice. This can be made precise with holonomy.

Identify Möbius bundle with the quotient space of $\mathbb{R}^2$:

$$\text{Möb} := \mathbb{R}^2/\sim, \qquad (x, y) \sim (x + 1, -y).$$

Every leaf is diffeomorphic to $\mathbb{S}^1$, whose fundamental group $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ is generated by a loop of it with winding number 1. The holonomy group of a leaf is thus a quotient of $\mathbb{Z}$. For every other leaf L except for the middle one and choose $x \in L$. There exists a transverse section at x which does not cross the middle leaf L_{mid}. Those leaves which intersect the transverse section are leaves of a trivial foliation of $\mathbb{S}^1 \times I$ for some open interval $I \subseteq \mathbb{R}$ describing this transverse section. Thus $\mathrm{Hol}(L) = 0$.

The middle leaf L_{mid} is the image of the line $y = 0$ under the quotient map $\mathbb{R}^2 \to \text{Möb}$. Choose the basepoint to be $(-1, 0) \in L_{\mathrm{mid}}$ and the transverse section be the line $x = -1$. Then $\pi_1(L_{\mathrm{mid}})$ is generated by the straightline path α connecting x and $x' = (1, 0)$, which becomes a loop of winding number 1 in Möb. Now for any other point $(-1, -\varepsilon)$ in an open neighbourhood of $(-1, 0)$, the holonomy map over α sends $(-1, -\varepsilon)$ to $(1, -\varepsilon) \sim (-1, \varepsilon)$. This means that the generator of $\pi_1(L_{\mathrm{mid}})$ is sent to a generator of degree 2. This implies that $\mathrm{Hol}(L_{\mathrm{mid}}) = \mathbb{Z}/2$.

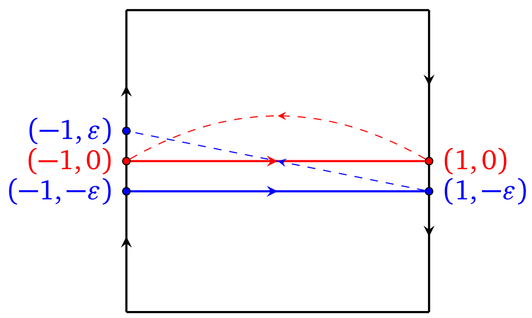


Figure 2.2: Holonomy of the middle leaf of the Möbius band

Example 2.9 (Reeb foliation). Consider the Reeb foliation of the solid torus as in Example 1.3.6. All non-compact leaves (which are diffeomorphic to $\mathbb{R}^2$) are contractible and have trivial holonomy group. The interesting case is the unique compact leaf $L = \mathbb{T}^2$. Identify a transverse section at $x \in L$ with $[0, +\infty)$ where x is identified with 0. Then $\mathrm{Hol}(L)$ is generated by $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ and $f(t) < t$ for all $t > 0$.

Likewise, the Reeb foliation of the unique compact leaf L of $\mathbb{S}^3$ has holonomy group $\mathbb{Z}^2$ generated by

$$f : \mathbb{R} \to \mathbb{R}, \qquad \begin{cases} f(t) = t, & t \leq 0; \\ f(t) < t, & t > 0. \end{cases}$$

$$h : \mathbb{R} \to \mathbb{R}, \qquad \begin{cases} h(t) = t, & t \geq 0; \\ h(t) > t, & t < 0. \end{cases}$$

Here the transverse section at $x \in L$ is identified with $\mathbb{R}$.

2.2 Stability

Definition 2.10 ([22, Definition 4.2])**.** Let $(M, \mathscr{F})$ be a foliated manifold. A subset $B \subseteq M$ is called *stable*, if for every open neighbourhood W of B in M, there exists an open neighbourhood $W' \subseteq W$ of B in M, such that every leaf intersecting W' is contained in W.

One of the classical problems in foliation theory is to find necessary and sufficient conditions for a leaf $L \subseteq M$ to be stable. These are given by stability theorems. First of all, there is a close relation between stability and finiteness of the holonomy group.

Example 2.11. Consider the foliation of $\mathbb{R}^2 \setminus \{(0,0)\}$ by cocentred circles $\{(x,y) \mid x^2 + y^2 = r\}$. Then each leaf is stable and compact, and each leaf has trivial holonomy group.

Example 2.12. The Reeb foliation of either the solid torus or of $\mathbb{S}^3$ has a unique compact leaf. It is unstable, and has infinite holonomy group.

Indeed, we have the following sufficient condition.

Theorem 2.13 ([22, Theorem 4.2])**.** *A compact leaf with finite holonomy group is stable.*

Idea of proof. Let L be a compact leaf with finite holonomy group $\mathrm{Hol}(L)$. Choose a basepoint $x \in L$ and a transverse section Σ at x. The following proof is an excerpt from the reference. I might complete it at some point.

- By compactness, one can find a finite set of foliation charts $\{U_1, \ldots, U_k\}$ covering the leaf.
- Since $\mathrm{Hol}(L, x)$ is finite and $\{U_1, \ldots, U_k\}$ is a finite cover, we are able to shrink the domain, so that all holonomy maps h_j are defined on a common open subset of the transverse section Σ.
- Take the union of all plaques of the form

  $$C_y^* = \{\pi_j^{-1}(h_j(f_{\beta_i}(y))) \mid 1 \leq i \leq k,\ 1 \leq j \leq r\}$$

  Then one can show that C_y^* is the leaf containing y.
- Finally, set

  $$W' = \bigcup_{y \in D'} F_y \subseteq W.$$

  This is the set required in the definition of stability. □

2.2.1 Riemannian foliation

Let M be an n-dimensional manifold. Denote by $\mathfrak{X}(M)$ the space of smooth vector fields over M. A symmetric, $\mathrm{C}^\infty(M)$-bilinear form

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathrm{C}^\infty(M)$$

is said to be *positive*, if

$$g(X, X) \geq 0, \qquad \text{for all } X \in \mathfrak{X}(M).$$

Let $x \in M$. Set

$$\ker g_x := \{\xi \in T_x M \mid g_x(\xi, T_x M) = 0\}.$$

The *Lie derivative* $\mathcal{L}_X g$ of g in the direction of $X \in \mathfrak{X}(M)$ is the symmetrm C^∞-bilinear form given by

$$\mathcal{L}_X g(Y,Z) := Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]).$$

Definition 2.14. Let $(M, \mathscr{F})$ be a foliated manifold. A *transverse metric* on M is a positive, symmetric $C^\infty(M)$-bilinear form g such that:

1. $\ker g_x = T_x\mathscr{F}$ for all $x \in M$.
2. $\mathcal{L}_X g = 0$ for any vector field X on M which is tangent to $\mathscr{F}$.

A *Riemannian foliation* $(\mathscr{F}, g)$ of a manifold M is a foliation $\mathscr{F}$ of M together with a transverse metric g.

Remark 2.15. Note that

1. 1 in the previous definition implies thast g is the pullback of a Riemannian structure on the normal bundle $N\mathscr{F}$, along the projection $TM \to N\mathscr{F}$.
2. 2 in the previous definition is a local condition: consider a foliation chart $(U, \varphi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ and set $\varphi(x) = (x_1, \dots, x_{n-q}, y_1, \dots, y_q)$. From 1 we know that g is determined by $g_{ij} := g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$. One can show, after some computation, that 2 holds iff $\frac{\partial g_{ij}}{\partial x_k} = 0$ for all k, i, j. Equivalently, $g|_U$ is the pullback of a Riemannian metric on $\mathbb{R}^2$ along the projection $\mathrm{pr}_{\mathbb{R}^q} \circ \varphi\colon U \to \mathbb{R}^q$.

In fact, we have:

Theorem 2.16. *Let $(M, \mathscr{F})$ be a foliated manifold. Then a Riemannian structure on $N\mathscr{F}$ determines a transverse metric iff it is holonomy invariant.*

Theorem 2.17 ([14, Theorem 2.6])**.** *Let $(\mathscr{F}, g)$ be a Riemannian foliation of M and assume that all leaves are compact. Then each leaf has finite holonomy group.*

Idea of proof. Let L be a leaf of $\mathscr{F}$, $x \in L$, and S a transverse section at x. The induced Riemannian structure of S yields an exponential map

$$\exp_x\colon B(0,\varepsilon) \to S, \quad \text{such that } \exp_x(0) = x,$$

where $B(0,\varepsilon)$ is the ε-ball centred at 0 of $T_x M$. Denote the image of $\exp_x$ by U.

Since L is compact, $\pi_1(L)$ is finitely generated, and hence $\mathrm{Hol}(L)$ is finitely generated, say, by holonomy maps $\{h_1, \dots, h_n\}$ where $h_i\colon V_i \to U$ is a diffeomorphism onto the image. We may replace all V_i's by a common domain $V := \cap_{i=1}^n V_i$. By Theorem 2.16, every h_i is an isometry. Thus we may assume that $V = \exp_x(B(0,\delta))$ for some $\delta \le \varepsilon$ and that $h_i(V) = V$ for all i. Then we have represented $\mathrm{Hol}(L)$ as a group of isometry on V.

Since L is compact, $L \cap S$ is a discrete set. But U, and hence V, is relatively compact in S. Thus $L \cap V \subseteq L \cap S$ is finite. Therefore the orbit of any element in $\mathrm{Hol}(L)$, viewed as an action of a group of isometry, is finite. The differential of this representation at x is a group of orthogonal linear transformations on $T_x V$, such that every orbit of the action is finite. This forces the group $\mathrm{Hol}(L)$ to be finite. □

Corollary 2.18. *Let $(\mathscr{F}, g)$ be a Riemannian foliation of M. If all leaves are compact, then all leaves are stable.*

2.2.2 Reeb stability

Finally, we have the following Reeb stability theorems as consequences of Theorem 2.13.

Theorem 2.19 (Local Reeb stability)**.** *Let $(M, \mathscr{F})$ be a foliated manifold. Let L be a compact leaf with finite holonomy group. Then for each neighbourhood W of L, there is an $\mathscr{F}$-invariant tubular neighbourhood $W' \subseteq W$ with*

$$\pi : W' \to L$$

satisfying:

1. *Every leaf $L' \subseteq W'$ is compact with finite holonomy group, hence stable.*
2. *If $L' \subseteq W'$ is a leaf. Then the restriction $\pi|_{L'} : L' \to L$ is a finite covering map.*

Theorem 2.20 (Global Reeb stability)**.** *Let $\mathscr{F}$ be a codimension-1 foliation of a compact connected manifold M. If $\mathscr{F}$ admits a compact leaf L with finite fundamental group, then all leaves of $\mathscr{F}$ are compact with finite fundamental group. So all leaves have finite holonomy group and hence stable.*

If $\mathscr{F}$ is transversely orientable, that is, $N\mathscr{F} = TM/T\mathscr{F}$ is orientable. Then every leaf of $\mathscr{F}$ is diffeomorphic to L, and M is the total space of a fibration $f : M \to \mathbb{S}^1$ with fibre L.

September 26, 2023

Lie groupoids

Speaker: Jack Ekenstam (Leiden University)

A *Lie groupoid* is a topological groupoid whose arrow space and unit space are equipped with smooth structures. A lot of constructions of topological groupoids (c.f. [16]) are therefore available, too. In this lecture, we will review these constructions and specify them to Lie groupoids. We will also introduce two Lie groupoids constructed from a foliation: the monodromy groupoid and the holonomy groupoid.

3.1 Definition of Lie groupoids

Recall that a groupoid can be defined as a small category whose arrows are invertible. We may specify the two sets $\mathscr{G}^{(1)}$ ("the set of arrows") and $\mathscr{G}^{(0)}$ ("the set of units"), together with several structure maps, to give the structure of a groupoid.

Definition 3.1 (Groupoids)**.** A groupoid $\mathscr{G}$ is given by two sets $\mathscr{G}^{(1)}$ (the set of arrows) and $\mathscr{G}^{(0)}$ (the set of units), together with the following five structure maps:

- a pair of maps $r, s : \mathscr{G}^{(1)} \rightrightarrows \mathscr{G}^{(0)}$ called the *range* map and the *source* map;
- a map $\cdot : \mathscr{G}^{(2)} \to \mathscr{G}^{(1)}$, $(g,h) \mapsto gh$ called the *multiplication* map, where

$$\mathscr{G}^{(2)} := \mathscr{G}^{(1)} \underset{s,\mathscr{G}^{(0)},r}{\times} \mathscr{G}^{(1)} = \{(g,h) \in \mathscr{G}^{(1)} \times \mathscr{G}^{(1)} \mid s(g) = r(h)\};$$

- a map $()^{-1} : \mathscr{G}^{(1)} \to \mathscr{G}^{(1)}$, $g \mapsto g^{-1}$, called the *inverse* map;
- a map $1 : \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$, $x \mapsto 1_x$ called the *unit* map;

such that:

1. $s(hg) = s(g), \quad r(hg) = r(h);$
2. $k(hg) = k(hg);$

3. $1_{r(g)}g = g = g1_{s(g)}$;
4. $s(g^{-1}) = r(g), \quad t(g^{-1}) = s(g), \quad g^{-1}g = 1_{s(g)}, \quad gg^{-1} = 1_{r(g)}$.

In most cases, we will only specify the range and source maps r and s, as all other structure maps are easy to define. It is usually convenient to write $\mathscr{G}$ for $\mathscr{G}^{(1)}$, and identify $\mathscr{G}^{(0)}$ with the subset $1(\mathscr{G}^{(0)})$ of $\mathscr{G}$.

The following notations are more widely used in the operator algebra community in contrast to differential geometers:

Definition 3.2. Let $\mathscr{G}$ be a groupoid.

- Let $x, y \in \mathscr{G}^{(0)}$. We define the *source fibre at y* to be $\mathscr{G}_y := s^{-1}(y)$, the *range fibre at x* to be $\mathscr{G}^x := r^{-1}(x)$, and $\mathscr{G}_y^x := \mathscr{G}^x \cap \mathscr{G}_y$.
- Let $A, B \subseteq \mathscr{G}^{(0)}$. We define $\mathscr{G}_B := s^{-1}(\mathscr{G})$, $\mathscr{G}^A := r^{-1}(A)$ and $\mathscr{G}_B^A := \mathscr{G}^A \cap \mathscr{G}_B$.

Definition 3.3. Let $\mathscr{G}$ be a groupoid.

- Let $A \subseteq \mathscr{G}^{(0)}$. Then $\mathscr{G}_A^A \subseteq \mathscr{G}$ is a subgroupoid, called the restriction of $\mathscr{G}$ to A.
- Let $x \in \mathscr{G}^{(0)}$. Then $\mathscr{G}_x^x$ is a group, called the *isotropy group* at x.

A groupoid homomorphism $\mathscr{G} \to \mathscr{H}$ is a functor between these categories. More concretely,

Definition 3.4 (Groupoid homomorphisms)**.** A (strict) groupoid homomorphism $\phi\colon \mathscr{G} \to \mathscr{H}$ is a map such that $\phi \times \phi(\mathscr{G}^{(2)}) \subseteq \mathscr{H}^{(2)}$ and $\phi(gh) = \phi(g)\phi(h)$ for all $(g, h) \in \mathscr{G}^{(2)}$. It is an *isomorphism* if there exists another groupoid homomorphism $\psi\colon \mathscr{H} \to \mathscr{G}$ such that $\psi \circ \phi = \mathrm{id}_{\mathscr{H}}$ and $\phi \circ \psi = \mathrm{id}_{\mathscr{G}}$.

A groupoid homomorphism $\phi : \mathscr{G} \to \mathscr{H}$ defines a pair of maps $\phi^{(1)}\colon \mathscr{G}^{(1)} \to \mathscr{H}^{(1)}$ and $\phi^{(0)}\colon \mathscr{G}^{(0)} \to \mathscr{H}^{(0)}$ in an obvious way.

Definition 3.5 (Lie groupoids)**.** A *Lie groupoid* is a groupoid $\mathscr{G}$, such that:

- $\mathscr{G}^{(0)}$ is a smooth, Hausdorff and second countable manifold.
- $\mathscr{G}^{(1)}$ is a smooth, not necessarily Hausdorff, not necessarily second countable manifold.
- $s\colon \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$ is a submersion with Hausdorff fibres.
- All the structure maps are smooth.

Remark 3.6. The definition above has the following immediate consequences:

1. The map $r\colon \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$ is also a submersion, because $r = s \circ ()^{-1}$, and $()^{-1}$ is a diffeomorphism as $(()^{-1})^2 = \mathrm{id}$.
2. $\mathscr{G}^{(2)}$ is also a smooth manifold because s is a submersion.

Definition 3.7 (Lie groupoid homomorphisms)**.** A homomorphism $\phi : \mathscr{G} \to \mathscr{H}$ between Lie groupoids is a groupoid homomorphism such that that $\phi^{(1)}\colon \mathscr{G}^{(1)} \to \mathscr{H}^{(1)}$ and $\phi^{(0)}\colon \mathscr{G}^{(0)} \to \mathscr{H}^{(0)}$ are smooth.

Example 3.8.

1. A *smooth manifold M* can be viewed as a Lie groupoid with $\mathscr{G}^{(1)} = \mathscr{G}^{(0)} = M$, $r = s = \mathrm{id}$.
2. A *Lie group G* can be viewed as a Lie groupoid with $\mathscr{G}^{(1)} = G$ and $\mathscr{G}^{(0)} = \mathrm{pt}$, r and s being the unique smooth map to a point.
3. The *pair groupoid* of a smooth manifold M is the groupoid $\mathrm{Pair}(M)$ with $\mathrm{Pair}(M)^{(1)} = M \times M$, $\mathrm{Pair}(M)^{(0)} = M$, $r = \mathrm{pr}_1$, $s = \mathrm{pr}_2$, $(x, y) \cdot (y, z) = (x, z)$.
4. *Equivalence relations*. Let $\mathscr{R}$ be an immersed submanifold of $M \times M$ that defines an equivalence relation. Then $\mathscr{R}$ defines a Lie groupoid with $\mathscr{G}^{(1)} = \mathscr{R}$, $\mathscr{G}^{(0)} = M$, $r = \mathrm{pr}_1$, $s = \mathrm{pr}_2$, $(x, y) \cdot (y, z) = (x, z)$. This is a Lie subgroupoid of $\mathrm{Pair}(M)$, meaning that there is an injective Lie groupoid homomorphism $\mathscr{R} \hookrightarrow \mathrm{Pair}(M)$.

5. *Action groupoid*. Let G be a Lie group acting smoothly on a manifold M (on the left), then the associated action groupoid $G \ltimes M$ is defined by $(G \ltimes M)^{(1)} = G \times M$, $(G \ltimes M)^{(0)} = M$, $s(g,x) = x$, $r(g,x) = gx$, $(g,hx) \cdot (h,x) = (gh,x)$.
6. *Kernel groupoid* or *submersion groupoid*. Any smooth map $p\colon N \to M$ induces a Lie groupoid homomorphism $p\colon \mathrm{Pair}(N) \to \mathrm{Pair}(M)$ in an obvious way. If p is moreover a submersion, then set $\ker p := \{(y,y') \in N \times N \mid p(y) = p(y')\}$. This defines a Lie groupoid with $\mathscr{G}^{(1)} = \ker p$, $\mathscr{G}^{(0)} = N$, $r = \mathrm{pr}_1$ and $s = \mathrm{pr}_2$. Note that $\ker p$ is a smooth manifold because p is a submersion.
7. *Fundamental groupoid*. Let M be a manifold. Its fundamental groupoid $\Pi_1(M)$ is defined as follows. The unit space of $\Pi_1(M)$ is M. An arrow from x to y is a homotopy class of paths in M, relative to endpoints, from x to y. The range (resp. source) map send a path to its endpoint (resp. starting point), and the multiplication is defined by concatenation of paths. The smooth structure of $\Pi_1(M)$ is a bit tricky, and we leave that to Section 3.2 of monodromy and holonomy groupoids.

Definition 3.9. Let $\mathscr{G}$ be a Lie groupoid.

1. A *global bisection* is a section $\sigma\colon \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$ of $s\colon \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$, such that $r \circ \sigma\colon \mathscr{G}^{(0)} \to \mathscr{G}^{(0)}$ is a diffeomorphism.

   The global bisections form a group called the *gauge group* of $\mathscr{G}$: the group laws are given by

$$\sigma\sigma'(x) := \sigma(r(\sigma'(x)))\sigma'(x), \qquad \sigma^{-1}(x) := \sigma(x)^{-1}.$$

2. A *local bisection* is a *local* section $\sigma\colon U \to \mathscr{G}^{(1)}$ of $s\colon \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$, where $U \subseteq \mathscr{G}^{(0)}$ is an open subset, such that $r \circ \sigma\colon U \to \mathscr{G}^{(0)}$ is an open embedding.

   The germs of local bisections form a groupoid $\mathrm{Bis}(\mathscr{G})$. Its multiplication is, locally, given by the same as the gauge group of global bisections.

Remark 3.10. It is not clear to me what the Lie groupoid structure $\mathrm{Bis}(\mathscr{G})$ is. Here is a possible construction. The collection $\mathrm{Diff}(M)$ of local diffeomorphisms of a manifold has the structure of a pseudogroup (a group-like object, satisfying some sheaf-like conditions; the prototype is precisely the set of local diffeomorphisms). Pseudogroups have a close relation with groupoids, namely, the germs of $\mathrm{Diff}(M)$ form naturally a groupoid over M: let f be a local diffeomorphism with $f(x) = y$. Then its representing germ of local diffeomorphisms from x to y is an element in this groupoid with source x and range y.

Let $\mathscr{G}$ be a Lie groupoid and σ be a local bisection. Since $r \circ \sigma$ is an open embedding, it is a local diffeomorphism of $\mathscr{G}^{(0)}$. The collection of all $r \circ \sigma$ with σ a local bisection is a subpseudogroup of $\mathrm{Diff}(\mathscr{G}^{(0)})$, and the germs define a groupoid over $\mathscr{G}^{(0)}$. What is the Lie groupoid structure? This is done by assuming the range and source maps are local diffeomorphisms. Since $\mathscr{G}^{(0)}$ carries a smooth structure, the charts pull back to charts of $\mathrm{Bis}(\mathscr{G})$ and in turn give a Lie groupoid.

Proposition 3.11. *The natural groupoid homomorphism* $\mathrm{Bis}(\mathscr{G}) \to \mathscr{G}$ *is smooth.*

Proposition 3.12. *Let $\mathscr{G}$ be a Lie groupoid. Then for every $g \in \mathscr{G}^{(1)}$, there exists a local bisection $\sigma\colon U \to \mathscr{G}^{(1)}$ such that $g \in \mathrm{Im}(\sigma)$.*

Proof. See [14, Proposition 5.3]. □

Definition 3.13 (Étale Lie groupoids)**.** A Lie groupoid $\mathscr{G}$ is called *étale* if $\dim \mathscr{G}^{(1)} = \dim \mathscr{G}^{(0)}$.

The word "étale" comes from the fact that if $\dim \mathscr{G}^{(1)} = \dim \mathscr{G}^{(0)}$, then the range and source maps s and r are local diffeomorphisms (because they are submersions of equal rank), and a local diffeomorphism is also called an étale map.

Example 3.14. It is is tautological that the groupoid of bisections $\mathrm{Bis}(\mathscr{G})$ is étale.

3.2 Groupoids of foliations

Definition 3.15. Let $(M, \mathscr{F})$ be a foliated manifold.

1. The *monodromy groupoid* $\mathrm{Mon}(M, \mathscr{F})$ is given by:

   - The unit space is M.
   - If x and y are two points on the same leaf L, then an arrow from x to y is a *homotopy* class of paths from x to y in L relative to the basepoints. That is, a homotopy of paths in L with basepoints fixed.

     If x and y are on different leaves, then there is no arrow from x to y.

2. The *holonomy groupoid* $\mathrm{Hol}(M, \mathscr{F})$ is defined in a similar fashion with $\mathrm{Mon}(M, \mathscr{F})$. The only difference is to replace the *homotopy* classes of paths by *holonomy* classes of paths.

Proposition 3.16. *Both* $\mathrm{Mon}(M, \mathscr{F})$ *and* $\mathrm{Hol}(M, \mathscr{F})$ *are étale Lie groupoids.*

Proof. See [14, Proposition 5.6]. □

In particular, let $\mathscr{F}$ be the trivial foliation of codimension 0 of a manifold M. Then $\mathrm{Mon}(M, \mathscr{F}) = \Pi_1(M)$. So $\Pi_1(M)$ is an étale Lie groupoid as well.

Proposition 3.17. *Let* $(M, \mathscr{F})$ *be a foliated manifold, with monodromy groupoid* $\mathrm{Mon}(M, \mathscr{F})$ *and holonomy groupoid* $\mathrm{Hol}(M, \mathscr{F})$*. We have the following immediate results:*

1. *There is a natural map* $\mathrm{Mon}(M, \mathscr{F}) \to \mathrm{Hol}(M, \mathscr{F})$*, by sending the homotopy class of a path to its holonomy class.*
2. *The orbits of* $\mathrm{Mon}(M, \mathscr{F})$ *and* $\mathrm{Hol}(M, \mathscr{F})$*, namely, the equivalence classes of points which are connected by an element in these groupoids, are the leaves of* $\mathscr{F}$*.*
3. *The isotropy groups of both of* $\mathrm{Mon}(M, \mathscr{F})$ *and* $\mathrm{Hol}(M, \mathscr{F})$ *are discrete. This is because the* M *is a manifold, whose fundamental group is discrete.*
4. *Let* $x \in L$*. The the range map of* $\mathrm{Mon}(M, \mathscr{F})$ *restricts to a universal cover* $\mathrm{Mon}(M, \mathscr{F})_x \to L$*.*

3.3 Constructing new Lie groupoids from old ones

3.3.1 Induced groupoids

Let $\mathscr{G}$ be a Lie groupoid, and $\phi : M \to \mathscr{G}^{(0)}$ be a smooth map. We are going to define a groupoid $\phi^*\mathscr{G}$ over M called the *induced groupoid* or *pullback groupoid*. An arrow in this groupoid from x to y should be identified with an arrow in $\mathscr{G}$ from $\phi(x)$ to $\phi(y)$. That means, we define

$$(\phi^*\mathscr{G})^{(1)} := M \underset{\phi, \mathscr{G}^{(0)}, r}{\times} \mathscr{G}^{(1)} \underset{s, \mathscr{G}^{(0)}, \phi}{\times} M$$

and equip it with the multiplication in $\mathscr{G}$. In other words, we define $(\phi^*\mathscr{G})^{(1)}$ to be the object constructed from two pullback diagrams:

$$\begin{array}{ccccc}
(\phi^*\mathscr{G})^{(1)} & & \longrightarrow & & M \\
\downarrow & & & & \downarrow{\scriptstyle \phi} \\
\mathscr{G}^{(1)} \underset{s, \mathscr{G}^{(0)}, \phi}{\times} M & \xrightarrow{\mathrm{pr}_1} & \mathscr{G}^{(1)} & \xrightarrow{r} & \mathscr{G}^{(0)} \\
\downarrow & & \downarrow{\scriptstyle s} & & \\
M & \xrightarrow[\phi]{} & \mathscr{G}^{(0)} & &
\end{array}$$

Note that the fibred product

$$\mathscr{G}^{(1)} \underset{s,\mathscr{G}^{(0)},\phi}{\times} M := \{(g,x) \in \mathscr{G}^{(1)} \times M \mid s(g) = \phi(x)\}$$

has a smooth structure because s is a submersion. Suppose that in addition $r \circ \mathrm{pr}_1$ is also a submersion, then $(\phi^*\mathscr{G})^{(1)}$ has a smooth structure as well. This construction is referred to as the *blow-up groupoid* in [16].

3.3.2 Smooth transformations

If we view a groupoid as a category, then a groupoid homomorphism $\mathscr{G} \to \mathscr{H}$ is just a functor. Then we may also speak about natural transformations, which are morphisms between functors. In the category of Lie groupoids, those transformations are required to be smooth as well.

More precisely, let $\phi, \psi \colon \mathscr{G} \rightrightarrows \mathscr{G}$ be Lie groupoid homomorphisms. A *smooth (natural) transformation* $T \colon \phi \Rightarrow \psi$ is a smooth map

$$T \colon \mathscr{G}^{(0)} \to \mathscr{H}^{(1)},$$

such that:

- for every $x \in \mathscr{G}^{(0)}$, $T(x)$ is an arrow from $\phi(x)$ to $\psi(x)$;
- for every arrow g from x to y, the diagram

$$\begin{array}{ccc} \phi(x) & \xrightarrow{T(x)} & \psi(x) \\ {\scriptstyle \phi(g)}\downarrow & & \downarrow{\scriptstyle \psi(g)} \\ \phi(y) & \xrightarrow[T(y)]{} & \psi(y) \end{array}$$

  which in other words means that T is a natural transformation if we forget the smooth structures of $\mathscr{G}$ and $\mathscr{H}$.

In particular, given Lie groupoids $\mathscr{G}$ and $\mathscr{H}$, we may build a new groupoid, whose:

- objects are homomorphisms $\mathscr{G} \to \mathscr{H}$;
- an arrow from ϕ to ψ is a smooth natural transformation $T \colon \phi \Rightarrow \psi$.

In fact, Lie groupoids, homomorphisms and transformations form a 2-category.

3.3.3 Products and sums

In the 2-category of Lie groupoids described above, the 2-product and 2-coproduct (i.e. the corresponding universal objects in a 2-category) can be explicitly constructed as the product Lie groupoid and the direct sum Lie groupoid. Given Lie groupoids $\mathscr{G}$ and $\mathscr{H}$, we may define:

- The product groupoid $\mathscr{G} \times \mathscr{H}$, with $(\mathscr{G} \times \mathscr{H})^{(1)} = \mathscr{G}^{(1)} \times \mathscr{H}^{(1)}$ and $(\mathscr{G} \times \mathscr{H})^{(0)} = \mathscr{G}^{(0)} \times \mathscr{H}^{(0)}$. The structure maps are inherited from $\mathscr{G}$ and $\mathscr{H}$.
- The direct sum groupoid $\mathscr{G} \oplus \mathscr{H}$, with $(\mathscr{G} \oplus \mathscr{H})^{(1)} = \mathscr{G}^{(1)} \coprod \mathscr{H}^{(1)}$ and $(\mathscr{G} \times \mathscr{H})^{(0)} = \mathscr{G}^{(0)} \coprod \mathscr{H}^{(0)}$. The structure maps are inherited from $\mathscr{G}$ and $\mathscr{H}$.

3.3.4 Strong fibred products

Let $\phi\colon \mathscr{G} \to \mathscr{K}$ and $\psi\colon \mathscr{H} \to \mathscr{K}$ be Lie groupoid homomorphisms. We may define a groupoid $\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H}$, called the *strong fibred product*, by

$$(\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H})^{(1)} := \mathscr{G}^{(1)} \underset{\phi^{(1)},\mathscr{K}^{(1)},\psi^{(1)}}{\times} \mathscr{H}^{(1)} = \{(g,h) \in \mathscr{G}^{(1)} \times \mathscr{H}^{(1)} \mid \phi^{(1)}(g) = \psi^{(1)}(h)\};$$
$$(\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H})^{(0)} := \mathscr{G}^{(0)} \underset{\phi^{(0)},\mathscr{K}^{(0)},\psi^{(0)}}{\times} \mathscr{H}^{(0)} = \{(x,y) \in \mathscr{G}^{(0)} \times \mathscr{H}^{(0)} \mid \phi^{(0)}(x) = \psi^{(0)}(y)\}.$$

The structure maps are obvious. This gives a groupoid, which is not always a Lie groupoid: it is, if the map $\phi^{(0)} \times \psi^{(0)}\colon \mathscr{G}^{(0)} \times \mathscr{H}^{(0)} \to \mathscr{K}^{(0)} \times \mathscr{K}^{(0)}$ is transverse to the diagonal

$$\Delta\mathscr{K}^{(0)} = \{(x,x) \mid x \in \mathscr{K}^{(0)}\}.$$

In that case we have that

$$(\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H})^{(0)} = (\phi^{(0)} \times \psi^{(0)})^{-1}(\Delta\mathscr{K}^{(0)})$$

is indeed a manifold.

3.3.5 Weak fibred products

As opposed to strong fibred products, a "larger" and in fact universal fibred product can be constructed as follows. Let $\phi\colon \mathscr{G} \to \mathscr{K}$ and $\psi\colon \mathscr{H} \to \mathscr{K}$ be Lie groupoid homomorphisms. The *weak fibred product P* is defined as follows:

- Objects are triples (x,k,y) with $x \in \mathscr{G}^{(0)}$, $y \in \mathscr{H}^{(0)}$ and $k \in \mathscr{K}_{\phi(x)}^{\psi(y)}$.
- An arrow from (x,k,y) to (x',k',y') is a pair (g,h) with $g \in \mathscr{G}^{(1)}$, $h \in \mathscr{H}^{(1)}$ such that the diagram

  $$\begin{array}{ccc} \phi(x) & \xrightarrow{\ k\ } & \psi(y) \\ {\scriptstyle \phi(g)}\big\downarrow & & \big\downarrow{\scriptstyle \psi(h)} \\ \phi(x') & \xrightarrow[\ k'\ ]{} & \psi(y') \end{array}$$

  commutes in $\mathscr{K}$ (viewed as a category).

  Remark: To my knowledge, P is usually called a comma category, denoted by $\phi \downarrow \psi$. A comma category can be understood as a 1-categorical fibred product or even a 2-categorical limit, in the 2-category of categories or groupoids. This explains why this construction is indeed universal.

The unit space of P can be also written as

$$P^{(0)} = \mathscr{G}^{(0)} \underset{\phi^{(0)},\mathscr{K}^{(0)},r}{\times} \mathscr{K}^{(1)} \underset{s,\mathscr{K}^{(0)},\psi}{\times} \mathscr{H}^{(0)},$$

which is not always a manifold. But if either $\phi^{(0)}\colon \mathscr{G}^{(0)} \to \mathscr{K}^{(0)}$ or $\psi^{(0)}\colon \mathscr{H}^{(0)} \to \mathscr{K}^{(0)}$ is a submersion, then $P^{(0)}$ has a manifold structure. And one can show that $P^{(1)}$ has a manifold structure as well. These render a Lie groupoid structure for P.

3.4 Equivalence of Lie groupoids

Recall from category theory that two categories $\mathscr{C}$ and $\mathscr{D}$ are *strongly equivalent*, if there exist a pair of functors

$$F\colon \mathscr{C} \to \mathscr{D}, \qquad G\colon \mathscr{D} \to \mathscr{C},$$

such that there are natural isomorphisms

$$F \circ G \simeq \mathrm{id}_{\mathscr{D}}, \qquad G \circ F \simeq \mathrm{id}_{\mathscr{C}} .$$

The functors F and G are called strong equivalences between the categories $\mathscr{C}$ and $\mathscr{D}$.

A well-known result in category theory states that, assuming the axiom of choice, then a functor $F: \mathscr{C} \to \mathscr{D}$ is a strong equivalence iff it is a *weak equivalence*, which means:

- F is *essentially surjective*, that is, for every $y \in \mathscr{D}$, there exists $x \in \mathscr{C}$ such that $Fx \simeq y$ in $\mathscr{D}$.
- F is *fully faithful*, that is, for every $x, y \in \mathscr{C}$, there is a natural bijection $\mathscr{D}(Fx, Fy) \simeq \mathscr{C}(x, y)$.

This section aims at adapting all these to Lie groupoid. This means we need to add extra smoothness conditions to the functors as well as the natural transformations, as follows.

Definition 3.18 (Strong equivalences)**.** A Lie groupoid homomorphism $\phi: \mathscr{G} \to \mathscr{H}$ is called a *strong equivalence*, if there is another Lie groupoid homomorphism $\psi: \mathscr{H} \to \mathscr{G}$, together with *smooth* transformations

$$T: \phi \circ \psi \Rightarrow \mathrm{id}_{\mathscr{H}}, \qquad S: \psi \circ \phi \Rightarrow \mathrm{id}_{\mathscr{G}} .$$

Definition 3.19 (Weak equivalences)**.** A Lie groupoid homomorphism $\phi: \mathscr{G} \to \mathscr{H}$ is called a *weak equivalence*, if:

- ϕ is *essentially surjective*: the map

  $$\mathscr{H}^{(1)} \underset{s, \mathscr{H}^{(0)}, \phi}{\times} \mathscr{G}^{(0)} \xrightarrow{\mathrm{pr}_1} \mathscr{H}^{(1)} \xrightarrow{r} \mathscr{H}^{(0)}$$

  is a surjective submersion. Here

  $$\mathscr{H}^{(1)} \underset{s, \mathscr{H}^{(0)}, \phi}{\times} \mathscr{G}^{(0)} := \{(h, x) \in \mathscr{H}^{(1)} \times \mathscr{G}^{(0)} \mid s(h) = \phi(x)\}.$$

- ϕ is *fully faithful*: the following diagram

  $$\begin{array}{ccc} \mathscr{G}^{(1)} & \xrightarrow{\phi^{(1)}} & \mathscr{H}^{(1)} \\ {\scriptstyle s \times t}\big\downarrow & & \big\downarrow{\scriptstyle s \times t} \\ \mathscr{G}^{(0)} \times \mathscr{G}^{(0)} & \xrightarrow[\phi^{(0)} \times \phi^{(0)}]{} & \mathscr{H}^{(0)} \times \mathscr{H}^{(0)} \end{array}$$

  is a fibred product (pullback diagram) in the category of manifolds.

Proposition 3.20. *Every strong equivalence of Lie groupoids is a weak equivalence.*

Remark 3.21. It is important to remark that, the existence of a weak equivalence $\phi: \mathscr{G} \to \mathscr{H}$ does *not* guarantee the existence of a weak equivalence $\psi: \mathscr{H} \to \mathscr{G}$. So when we say two Lie groupoids $\mathscr{G}$ and $\mathscr{H}$ are *weakly equivalent*, or *Morita equivalent*, we shall mean that they are equivalent in the equivalence relation *generated by* weak equivalences. That means, if there are weak equivalences

$$\phi: \mathscr{G} \to \mathscr{K}, \qquad \psi: \mathscr{H} \to \mathscr{K} .$$

Proposition 3.22. *A Lie groupoid is transitive iff it is Morita equivalent to a Lie group.*

October 17, 2023

C*-algebras of foliations

Speaker: Bram Mesland (Leiden University)

The C*-algebra of a foliation is, essentially, the groupoid C*-algebra of the holonomy groupoid of the foliation. Renault's work [21] provides a general theory of groupoid C*-algebras. For a locally compact, Hausdorff groupoid $\mathcal{G}$, one constructs a "fibrewise measure" called the Haar system, then complete the compactly supported functions $C_c(\mathcal{G})$ on this groupoid under a suitable norm using a representation based on the Haar system. There can be different choices for the Haar system, yet they yield the same C*-algebra. An account of this is in Yufan's talk on September 27, 2022 at the groupoid seminar [16].

A major difference of this lecture is that we face the subtlety of allowing groupoids to be non-Hausdorff. This is not rare in the study of foliations: the holonomy groupoid of a foliation is in general only a *locally Hausdorff* groupoid, which means that every point has a Hausdorff open neighbourhood, but fails to be Hausdorff. An elegant construction for the C*-algebras of non-Hausdorff groupoids is provided in [12], which will be described below.

4.1 C*-algebras of non-Hausdorff groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e.

$\mathcal{G}^{(1)}$ and $\mathcal{G}^{(0)}$ are topological spaces, and all structure maps are continuous.

Definition 4.1. We say a topological groupoid $\mathcal{G}$ is *locally Hausdorff*, if:

- $\mathcal{G}^{(1)}$ is locally compact and locally Hausdorff, i.e. every point in $\mathcal{G}^{(1)}$ has an open neighbourhood that is Hausdorff under its subspace topology, and a compact neighbourhood.
- $\mathcal{G}^{(0)}$ is locally compact and Hausdorff.

Remark 4.2. The definition above is taken from [12, Definition 1.1], but we removed the condition (d) and (e) due to the following reasons. As shown by Tu [23] (c.f. footnote remark of [12, Definition 1.1]), $\mathcal{G}_x$ is automatically Hausdorff for every $x \in \mathcal{G}^{(0)}$. The condition (e) is a corollary of the existence of a Haar system, see Lemma 4.9.

From now on, by a non-Hausdorff groupoid, or simply a groupoid $\mathcal{G}$ we shall always mean a *locally Hausdorff* topological groupoid. The main problem with defining the C*-algebra of a non-Hausdorff groupoid comes from the facts that:

1. A compact set of a non-Hausdorff space may not be closed.
2. A compactly supported continuous function on an open Hausdorff subset (whose existence is guaranteed by local Hausdorffness) may not be continuous on the whole space.

Example 4.3. Let $X := \{0, 1\}$ with the topology given by the collection of open sets $\{\emptyset, \{0\}, \{0, 1\}\}$. Namely, the singleton $\{0\}$ is open and $\{1\}$ is not. Then the set $\{0\}$ is compact, but not closed. Any function $f : X \to \mathbb{C}$ is specified by its value on 0 and 1. Let $f(0) = 1$ and $f(1) = 0$. Then $f^{-1}(\{0\}) = \{0\}$ is not closed, hence f is not continuous on X, but f is a continous function on the open set $\{0\}$ and has compact support in $\{0\}$.

The following one-line definition is given in [9]; see also [12]:

Definition 4.4. Let $\mathcal{G}$ be a locally Hausdorff groupoid. Define

$$C_c(\mathcal{G}) := \text{span}\{C_c(V) \mid V \subseteq \mathcal{G} \text{ is open and Hausdorff}\}.$$

Note that despite of the symbol, a function in $\mathrm{C_c}(\mathcal{G})$ may not be continuous on $\mathcal{G}$ at all.

Lemma 4.5 ([12, Lemma 1.3])**.** *Let $\{U_i\}$ be* any *open cover of $\mathcal{G}$ by Hausdorff sets. Then*

$$\mathrm{C_c}(\mathcal{G}) = \operatorname{span}\{\mathrm{C_c}(U_i)\}.$$

If $\mathcal{G}$ is moreover a Lie groupoid. Then

$$\mathrm{C_c}(\mathcal{G}) = \operatorname{span}\{\varphi \circ \chi \mid \chi : U \to \mathbb{R}^k \text{ is a Hausdorff local chart, } \varphi \in \mathrm{C_c}(\chi(U))\}.$$

Remark 4.6. The latter context also defines $\mathrm{C_c^\infty}(\mathcal{G})$. If $\mathcal{G}$ is Hausdorff, then it coincide with the usual definition of $\mathrm{C_c^\infty}(\mathcal{G})$ of a Lie groupoid.

We want to define Haar systems for non-Hausdorff groupoids. Fortunately, the usual definition (for a Hausdorff groupoid) works because:

Lemma 4.7. *If $\mathcal{G}$ is locally compact and $\mathcal{G}^{(0)}$ is Hausdorff, then for all $x \in \mathcal{G}^{(0)}$, $r^{-1}(x)$ and $s^{-1}(x)$ are Hausdorff.*

Definition 4.8. A *Haar system* on a non-Hausdorff groupoid $\mathcal{G}$ is a family of measures $\{\nu^x\}_{x\in\mathcal{G}^{(0)}}$ such that:

- $\operatorname{supp} \nu^x = r^{-1}(x)$.
- For any $f \in \mathrm{C_c}(\mathcal{G})$ (defined in Definition 4.4), the map

$$\mathcal{G}^{(0)} \to \mathbb{C}, \qquad x \mapsto \int_{\mathcal{G}} f(\xi)\,\mathrm{d}\nu^x(\xi)$$

  is continuous.
- For all $\eta \in \mathcal{G}$ and $f \in \mathrm{C_c}(\mathcal{G})$:

$$\int_{\mathcal{G}} f(\eta\xi)\,\mathrm{d}\nu^{r(\eta)}\xi = \int_{\mathcal{G}} f(\xi)\,\mathrm{d}\nu^{s(\eta)}\xi.$$

Lemma 4.9. *If $\mathcal{G}$ admits a Haar system. Then the range and source maps r, s are open.*

Let $\mathcal{G}$ be a non-Hausdorff groupoid which is equipped with a Haar system. Then $\mathrm{C_c}(\mathcal{G})$ is a *-algebra, with operations:

$$f * g(\eta) := \int f(\xi)g(\xi^{-1}\eta)\,\mathrm{d}\nu^{r(\eta)}\xi,$$
$$f^*(\eta) := \overline{f(\eta^{-1})}.$$

Moreover, we have a family of Hilbert spaces $\{L^2(\mathcal{G}, \nu^x)\}_{x\in\mathcal{G}^{(0)}}$ together with a family of representations

$$\pi_x : \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G}, \nu^x)), \qquad \pi_x(f)\varphi(\eta) := \int f(\xi)\varphi(\xi^{-1}\eta)\,\mathrm{d}\nu^{r(\eta)}\xi.$$

Definition 4.10. The *reduced* C^**-norm* $\|\cdot\|_{\mathrm{r}}$ on $\mathrm{C_c}(\mathcal{G})$ is

$$\|f\|_{\mathrm{r}} := \sup_{x\in\mathcal{G}^{(0)}} \|\pi_x(f)\|_{\mathbb{B}(L^2(\mathcal{G},\nu^x))}.$$

The *reduced groupoid* C^**-algebra* $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ is the completion of $\mathrm{C_c}(\mathcal{G})$ in the reduced C^*-norm.

Definition 4.11. A groupoid is *étale* if the range and source maps $r, s : \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ are local homeomorphisms.

Lemma 4.12. *If $\mathscr{G}$ is étale. Then for every $\eta \in \mathscr{G}$ and $f \in C_c(\mathscr{G})$: $|f(\eta)| \leq \|f\|_r$. In particular this implies that $C^*_r(\mathscr{G}) \subseteq C_0(\mathscr{G})$.*

Proof. Choose a bump function χ around $s(\eta)$ and such that it is 1 on the range fibre of η. Then

$$f(\eta) = \int f(\xi)\chi(\xi^{-1}\eta)\,\mathrm{d}\nu^{r(\eta)}\xi. \qquad \square$$

Example 4.13. Let H be a topological group which acts on a space X. Then the action groupoid

$$X \rtimes H := \{(x,h) \mid x \in X,\ h \in H\}$$

with range and source maps

$$r(x,h) := x, \quad s(x,h) := h^{-1}x$$

carries a Haar system, which on each fibre is just the Haar measure of H.

4.2 Foliation groupoids and C*-algebras

Given a foliated manifold $(M, \mathscr{F})$ of codimension-q, we are now going to define a Haar system for its holonomy groupoid $\mathrm{Hol}(M,\mathscr{F}) \rightrightarrows M$, which consists of the *holonomy* classes of paths in a leaf of M. This is a topological groupoid, and in fact a Lie groupoid, whose topology is the quotient topology of the path space

$$\mathrm{Path}(M,\mathscr{F}) := \{\gamma\colon [0,1] \to L \mid L \text{ is a leaf of } M\}$$

with the compact-open topology.

The foliation $\mathscr{F}$ of yields an involutive subbundle $T\mathscr{F}$, and one sets the normal bundle $N\mathscr{F} := TM/T\mathscr{F}$. Assume that M carries a Riemannian metric g. Then this yields a direct sum decomposition of vector bundles

$$TM = T\mathscr{F} \oplus N\mathscr{F}. \tag{4.1}$$

Let g be a Riemannian metric on M. Assume further that M is orientable, and $\mathscr{F}$ is *transversally orientable* which means that $N\mathscr{F}$ is orientable. A Haar system for $\mathrm{Hol}(M,\mathscr{F})$ can be obtained as follows[2]. Take the volume form of M, which is a non-vanishing top form $\omega \in \Omega^n(M) = \Gamma(M, \bigwedge^n T^*M)$. Since $\mathscr{F}$ is transversally orientable, there is another non-vanishing section $\sigma \in \Gamma(M, \bigwedge^q N^*\mathscr{F})$ where $N^*\mathscr{F}$ is the dual bundle of $N\mathscr{F}$. With respect to the decomposition (4.1), we have

$$\Omega^n(M) = \Gamma\left(M, \bigwedge^{n-q} T^*\mathscr{F} \otimes \bigwedge^q N^*\mathscr{F}\right)$$

and hence $\iota_\sigma\omega \in \Gamma(\bigwedge^{n-q} T^*\mathscr{F})$ is non-vanishing. This gives a Haar system $\{\lambda^x\}$ supported on leaves. For a different metric g' on M, the induced Haar system $\{\nu^x\}$ gives rise to a continuous, positive Radon–Nikodým derivative

$$x \mapsto \frac{\mathrm{d}\nu^x}{\mathrm{d}\lambda^x},$$

which can be used to construct an isomorphism between the *-algebras

$$C_c(\mathscr{G}, \lambda^x) \xrightarrow{\sim} C_c(\mathscr{G}, \nu^x),$$

and hence an isomorphism between the C*-algebras. This is reminiscent of the isomorphism between the groupoid C*-algebra of a group (viewed as a groupoid), and the group C*-algebra. See [16, Example 2.23].

Definition 4.14. A *transversal* of $\mathscr{G}$ is a closed subset $T \subseteq \mathscr{G}^{(0)}$ such that:

- T intersects every source fibre.

[2] I thank Stijn Velstra for pointing out my sloppy mistake and showing me the correct construction.

- The maps $r|_{\mathcal{G}_T} : \mathcal{G}_T \to \mathcal{G}^{(0)}$ and $s|_{\mathcal{G}_T} : \mathcal{G}_T \to T$ are open, where

$$\mathcal{G}_T := \{g \in \mathcal{G} \mid s(g) \in T\}$$

  is equipped with the relative topology.

Proposition 4.15. *The subgroupoid $\mathcal{G}_T^T$, which is the restriction of $\mathcal{G}$ to the transversal T, is Morita equivalent to $\mathcal{G}$.*

4.2.1 Recap: Morita equivalence

In the above we are using *Morita equivalence* of topological groupoids. This concept was discussed thoroughly in the groupoid seminar last year. I copied some of the definitions and results from [16] for convenience and completeness.

Definition 4.16. Let $\mathcal{G}$ be a groupoid. Let X be a space together with a continuous map $r_X : X \to \mathcal{G}^{(0)}$ called the *moment map*. A left action of $\mathcal{G}$ on X is a continuous map

$$\mathcal{G} \underset{s,\mathcal{G}^{(0)},r_X}{\times} X \to X, \qquad (\gamma, x) \mapsto \gamma x,$$

where

$$\mathcal{G} \underset{s,\mathcal{G}^{(0)},r_X}{\times} X := \{(\gamma, x) \in \mathcal{G} \times X \mid s(\gamma) = r_X(x)\},$$

such that

- $r_X(x)x = x$ for all $x \in X$.
- If $(\gamma, \eta) \in \mathcal{G}^{(2)}$ and $(\eta, x) \in \mathcal{G} \underset{s,\mathcal{G}^{(0)},r_X}{\times} X$. Then $(\gamma\eta, x) \in \mathcal{G} \underset{s,\mathcal{G}^{(0)},r_X}{\times} X$ and $(\gamma\eta)x = \gamma(\eta x)$.

A right action is defined similarly, while in that case a moment map is denoted by s_X for consistency. If $\mathcal{G}$ acts on X on the left (resp. right), we write $G \curvearrowleft X$ (resp. $X \curvearrowright G$) and call X a left (resp. right) $\mathcal{G}$-space.

Definition 4.17. Let X be a left $\mathcal{G}$-space and $x \in X$. The *orbit* of x is

$$\left\{ \gamma x \;\middle|\; (\gamma, x) \in \mathcal{G} \underset{s,\mathcal{G}^{(0)},r_X}{\times} X \right\}.$$

Denote by $\mathcal{G}\backslash X$ the space of orbits equipped with the quotient topology. If X is a right $\mathcal{G}$-space, we write $X/\mathcal{G}$ for the space of orbits.

Definition 4.18. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids with r and s open. A locally-compact Hausdorff space Z is called a $(\mathcal{G}, \mathcal{H})$-(Morita) equivalence, if the followings hold:

1. Z is a free and proper left $\mathcal{G}$-space and a free and proper right $\mathcal{H}$-space.
2. The left $\mathcal{G}$-action commutes with the right $\mathcal{H}$-action.
3. The moment map $r : Z \to \mathcal{G}^{(0)}$ is open and induces a homeomorphism $Z/\mathcal{H} \to \mathcal{G}^{(0)}$.
   $s : Z \to \mathcal{H}^{(0)}$ is open and induces a homeomorphism $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$.

Two groupoids $\mathcal{G}, \mathcal{H}$ are Morita equivalent if there is a $(\mathcal{G}, \mathcal{H})$-Morita equivalence. A key result here is that Morita equivalent groupoids yield Morita equivalent C*-algebras:

Definition 4.19. Let A and B be C^*-algebras. An *imprimitivity A,B-bimodule* is given by a right Hilbert B-module E which is simultaneously a left Hilbert A-module , and such that the B-valued inner product $\langle\cdot,\cdot\rangle_B$ is compatible with the A-valued inner product ${}_A\langle\cdot,\cdot\rangle$. That is,

$$\langle x, ay\rangle_B = \langle a^* x, y\rangle_B, \quad {}_A\langle x b^*, y\rangle = {}_A\langle x, yb\rangle, \quad {}_A\langle x, y\rangle z = x\langle y, z\rangle_B$$

for all $x, y, z \in E$, $a \in A$ and $b \in B$.

We say A and B are *Morita–Rieffel* equivalent, if there exists an imprimitivity bimodule between them.

Morita–Rieffel equivalent C^*-algebras have the same representation theory and hence the same K-theory. That is, let us write $\mathrm{Rep}(A)$ for the category of Hilbert spaces admitting non-degenerate left actions of A, with arrows unitary intertwiners. Then we have:

Proposition 4.20. *If A and B are Morita–Rieffel equivalent through imprimitivity bimodule E. Then the functor*

$$E \otimes_B - \colon \mathrm{Rep}(B) \to \mathrm{Rep}(A)$$

is an equivalence of categories.

Theorem 4.21 (Brown–Green–Rieffel)**.** *Let A and B be C^*-algebras. The followings are equivalent:*

1. *A is Morita–Rieffel equivalent to B.*
2. *There exists a full Hilbert B-module E such that $A \cong \mathbb{K}_B(E)$.*

If moreover A and B are σ-unital. Then both 1 and 2 are also equivalent to:

3. *$A \otimes \mathbb{K}$ and $B \otimes \mathbb{K}$ are isomorphic as C^*-algebras.*

Theorem 4.22. *Morita equivalent groupoids have Morita–Rieffel equivalent groupoid C^*-algebras.*

4.2.2 Properties and examples

Let $(M, \mathscr{F})$ be a foliated manifold. Write $\mathrm{C}^*(M, \mathscr{F}) := \mathrm{C}^*_r(\mathrm{Hol}(M, \mathscr{F}))$ and call it the C^*-algebra of the foliation. Here we collect some properties of $(M, \mathscr{F})$ which are reflected by $\mathrm{C}^*(M, \mathscr{F})$.

Proposition 4.23. *If $\mathrm{Hol}(M, \mathscr{F})$ is Hausdorff. Then:*

1. *$\mathrm{C}^*(M, \mathscr{F})$ is simple iff every leaf is dense.*
2. *$\mathrm{C}^*(M, \mathscr{F})$ has an injective irrep on a Hilbert space iff there is a dense leaf.*
3. *$\mathbb{K}$ is a quotient of $\mathrm{C}^*(M, \mathscr{F})$ iff all leaves of $\mathscr{F}$ are closed.*

Recall that a C^*-algebra A is simple iff it does not have closed ideals other than A and $\{0\}$.

Example 4.24.

1. Let M be foliated by points, then $\mathrm{Hol}(M, \mathscr{F}) = M \rightrightarrows M$ and $\mathrm{C}^*(M, \mathscr{F}) = \mathrm{C}_0(M)$.
2. Let M be foliated by a single leaf, then $\mathrm{Hol}(M, \mathscr{F}) = M \times M \rightrightarrows M$ the pair groupoid, and $\mathrm{C}^*(M, \mathscr{F}) = \mathbb{K}(L^2 M)$.
3. Let $\pi \colon M \twoheadrightarrow B$ be a fibration with typical fibre F. Foliate M by $\{\pi^{-1}(b)\}_{b \in B}$. Then $\mathrm{C}^*(M, \mathscr{F}) = \mathrm{C}_0(B, \mathbb{K}(L^2 F))$.
4. Let G be a Lie group acting freely and properly on M. Foliate M by the orbits. Then $\mathrm{C}^*(M, \mathscr{F}) = \mathrm{C}_0(M) \rtimes G$.

Part II

C^*-algebras of Bratteli diagrams and foliations of translation surfaces

An introduction is to be added.

List of talks

5. Translation surfaces and bi-infinite Bratteli diagrams (Olga Lukina, 24/10/2023)

   We start with two definitions of translation surfaces: a constructive one, and a descriptive one. After investigating on several prominent examples of translation surfaces, we introduce Bratteli diagrams including their bi-infinite and ordered variants. The construction of a translation surface will be based on the ordered path space of an ordered bi-infinite Bratteli diagram. The reference is [20, §1-4].

6. The surface associated with a bi-infinite Bratteli diagram (Dimitris Gerontogiannis, 31/10/2023)

   In this talk, we focus on the topology and measure theory of the path space of an ordered, bi-infinite Bratteli diagram, and later on the “leaves” or tail equivalence classes of the path space. Towards the end we define a suitable “gluing” of the path space using the partial order thereon, which yields a translation surface. A highlight is that this order can also detect the singularities of this surface. The reference is [20, §3-6].

List of symbols

The paper [20] of Putnam and Treviño is quite involved. Moreover it consists of a substantial number of symbols, making it even more difficult to read. For our convenience, a collection of symbols used in this paper is provided below. I have been trying to make it complete, but some symbols which are used only locally might be dropped out.

Bratteli diagrams

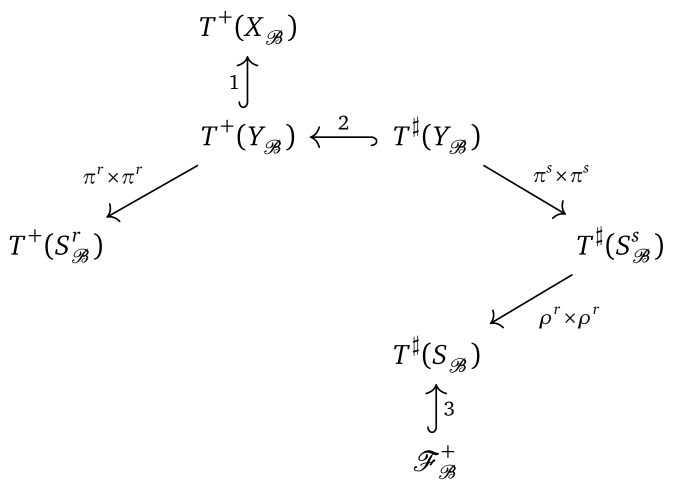
Let $\mathscr{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram with order $\leq_s$ and $\leq_r$ (5.12). Let $v \in V$ be a vertex and $x \in X_{\mathscr{B}}$ be a (bi-infinite) path of $\mathscr{B}$.

Symbol	Meaning	Reference
$X_{\mathscr{B}}$	Space of bi-infinite paths in $\mathscr{B}$.	5.10
$X_{\mathscr{B}}^{\pm}$	Space of left/right infinite paths in $\mathscr{B}$.	7.6
$X_v^{\pm}$	Space of left/right infinite paths in $\mathscr{B}$ starting/ending at v.	6.5
$X_{\mathscr{B}}^{s\text{-max}}, X_{\mathscr{B}}^{r\text{-max}}$, $X_{\mathscr{B}}^{s\text{-min}}, X_{\mathscr{B}}^{r\text{-min}}$	Space of s-maximal/r-maximal/ s-minimal/r-minimal paths in $\mathscr{B}$.	6.14
$X_{\mathscr{B}}^{\text{ext}}$	Extreme points of $X_{\mathscr{B}}$. $X_{\mathscr{B}}^{\text{ext}} := X_{\mathscr{B}}^{s\text{-max}} \cup X_{\mathscr{B}}^{r\text{-max}} \cup X_{\mathscr{B}}^{s\text{-min}} \cup X_{\mathscr{B}}^{r\text{-min}}$.	6.14
$\partial_s X_{\mathscr{B}}$, $\partial_r X_{\mathscr{B}}$	s-boundary of $X_{\mathscr{B}}$/r-boundary of $X_{\mathscr{B}}$. $\partial_s X_{\mathscr{B}} := \{x \in X_{\mathscr{B}} \mid x$ has an s-successor or an s-predecessor$\}$. $\partial_r X_{\mathscr{B}} := \{x \in X_{\mathscr{B}} \mid x$ has an r-successor or an r-predecessor$\}$.	6.15
$\partial X_{\mathscr{B}}$	Boundary of $X_{\mathscr{B}}$. $\partial X_{\mathscr{B}} := \partial_s X_{\mathscr{B}} \cap \partial_r X_{\mathscr{B}}$.	6.18
Δ_s, Δ_r	$\Delta_s : \partial_s X_{\mathscr{B}} \to \partial_s X_{\mathscr{B}}$ sending x to its s-successor or s-predecessor. $\Delta_r : \partial_r X_{\mathscr{B}} \to \partial_r X_{\mathscr{B}}$ sending x to its r-successor or r-predecessor.	6.17
$\Sigma_{\mathscr{B}}$	Singular points of $X_{\mathscr{B}}$. $\Sigma_{\mathscr{B}} := \{x \in \partial X_{\mathscr{B}} \mid \Delta_r \circ \Delta_s(x) \neq \Delta_s \circ \Delta_r(x)\}$.	6.18
$Y_{\mathscr{B}}$	$Y_{\mathscr{B}} := X_{\mathscr{B}} \setminus (X_{\mathscr{B}}^{\text{ext}} \cup \Sigma_{\mathscr{B}})$.	6.20
$S_{\mathscr{B}}^s$, $S_{\mathscr{B}}^r$	$S_{\mathscr{B}}^s := Y_{\mathscr{B}} \,/\, x \sim \Delta_s(x),\ x \in Y_{\mathscr{B}} \cap \partial_s X_{\mathscr{B}}$. $S_{\mathscr{B}}^r := Y_{\mathscr{B}} \,/\, x \sim \Delta_r(x),\ x \in Y_{\mathscr{B}} \cap \partial_r X_{\mathscr{B}}$.	7.1
$S_{\mathscr{B}}$	$S_{\mathscr{B}} := Y_{\mathscr{B}} \Big/ \begin{matrix} x \sim \Delta_s(x) \text{ if } x \in \partial_s X_{\mathscr{B}} \cap Y_{\mathscr{B}}; \\ x \sim \Delta_r(x) \text{ if } x \in \partial_r X_{\mathscr{B}} \cap Y_{\mathscr{B}}. \end{matrix}$	6.20
$T_N^+(x)$, $T_N^-(x)$	$T_N^+(x) := \{y \in X_{\mathscr{B}} \mid y_n = x_n$ for all $n > N\}$. $T_N^-(x) := \{y \in X_{\mathscr{B}} \mid y_n = x_n$ for all $n \leq N\}$.	6.9
$T^+(x)$, $T^-(x)$	Tail equivalence classes. $T^+(x) := \bigcup_{N \in \mathbb{Z}} T_N^+(x)$. $T^-(x) := \bigcup_{N \in \mathbb{Z}} T_N^-(x)$.	6.9
$x_{(n,+\infty)}$, $x_{[m,n)}$	$x_{(n,+\infty)} := (\ldots, x_{n+2}, x_{n+1})$. $x_{[m,n)} := (x_{n-1}, x_{n-2}, \ldots, x_m)$.	6.10
$E_{m,n}$	$E_{m,n} := \prod_{m<i\leq n} E_i = \{$Finite paths from V_m to $V_n\}$.	6.10
$E_{m,n}^Y$	Finite paths $p \in E_{m,n}$ with: - p is neither s-maximal, s-minimal, r-maximal nor r-minimal. - $X_{s(p)}^- p X_{r(p)}^+ \subseteq Y_{\mathscr{B}}$.	7.2

$E^s_{m,n}$	Set of pairs $p=(p_1,p_2)$ satisfying: - $p_i \in E^Y_{m,n}$. - p_2 is an s-successor of p_1.	7.23
$E^{s/r}_{m,n}$	Set of quadruples $p=(p_{1,1},p_{1,2},p_{2,1},p_{2,2})$ satisfying: - $p_{i,j} \in E^Y_{m,n}$. - $p_{i,j+1}$ is an s-successor of $p_{i,j}$. - $p_{i+1,j}$ is an r-successor of $p_{i,j}$.	7.3

Groupoids

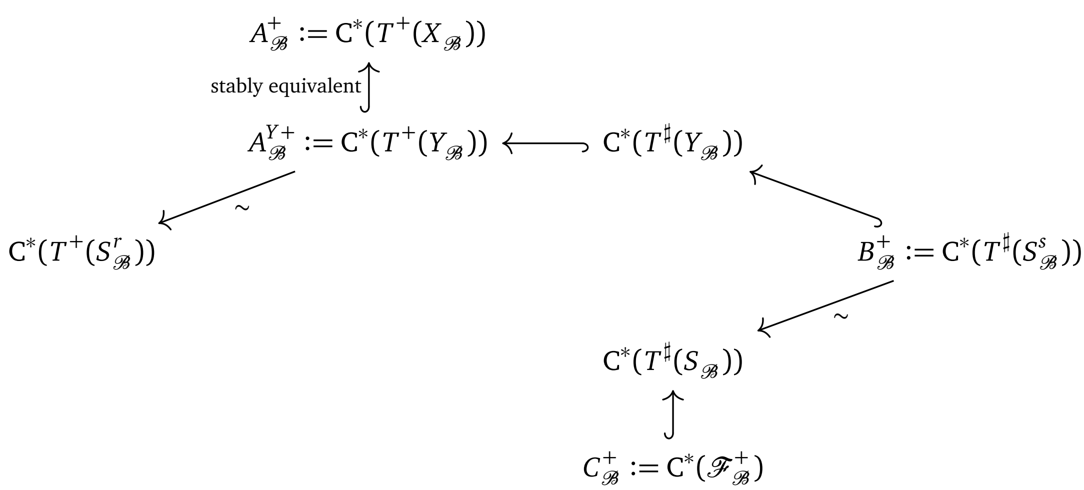
The groupoids in [20, §7] as well as their relations are listed below. By $\mathscr{G} \hookrightarrow \mathscr{H}$ I mean $\mathscr{G}$ is an open subgroupoid of $\mathscr{H}$.



1. $T^+(Y_{\mathscr{B}}) := T^+(X_{\mathscr{B}})\big|^{Y_{\mathscr{B}}}_{Y_{\mathscr{B}}}$.
2. $T^\sharp(Y_{\mathscr{B}}) := \big\{(x,y) \in T^+(Y_{\mathscr{B}}) \mid \text{If } x,y \in \partial_s X_{\mathscr{B}}, \text{ then } (\Delta_s(x),\Delta_s(y)) \in T^+(Y_{\mathscr{B}})\big\}$.
3. $\mathscr{F}^+_{\mathscr{B}} := \big\{(x,y) \in T^\sharp(S_{\mathscr{B}}) \mid x,y \text{ are in the same connected component}\big\}$.

C*-algebras

The groupoids in the diagram above yield their groupoid C*-algebras in the diagram below. Besides that, several inductive system are introduced in [20, §8], displaying these groupoid C*-algebras as AF-algebras and providing useful short exact sequences to compute their K-theory.



Inductive systems

Inductive system	Inductive limit	Reference
$A^{+}_{m,n}$	$A^{+}_{\mathscr{B}}$	7.15
$A^{Y+}_{m,n}$	$A^{Y+}_{\mathscr{B}}$	7.17
$AC^{+}_{m,n}$		7.18
$AC^{Y+}_{m,n}$		7.18
$B^{+}_{-n,n}$	$B^{+}_{\mathscr{B}}$	7.21
$C^{+}_{-n,n}$	$C^{+}_{\mathscr{B}}$	7.27
$G_{m,n}$		7.23
$H_{m,n}$		7.29

October 24, 2023

Translation surfaces and bi-infinite Bratteli diagrams

Speaker: Olga Lukina (Leiden University)

Translation surfaces are surfaces obtained by identifying several edges of polygons in the Euclidean plane. They are useful as models of dynamical systems on the unit interval $[0,1]$. The *Bratteli diagrams* are first used by operator algebraists for studying AF-algebras, but later endued with dynamical meanings after equipped with an order (on the path space).

An ordered Bratteli diagram therefore models a Cantor dynamical system. There is a well-known strategy to pass from a dynamical system on $[0,1]$ to one on the Cantor set by adding (countably many) limit points and equip the latter with an invariant measure. But for long it is unclear how to reverse this process. The recent paper [20] of Putnam and Treviño provides a solution.

In this first lecture, we provide their definitions and work out some practical examples, covering those from [20, §2–4].

5.1 Translation surfaces

A translation surface can be defined in various different ways. We start with a constructive definition.

Definition 5.1 (First definition, constructive)**.** Let $\mathscr{P}$ be an at most countable family of polygons in $\mathbb{R}^2$. For each polygon $P \in \mathscr{P}$, let $\mathscr{E}(P)$ denote the set of all line segments of the boundary of P. For each $e \in \mathscr{E}(P)$, denote the inward normal unit vector by n_e. Assume that:

There is a pairing (i.e. an involutive map)

$$f : \bigcup_{P\in\mathscr{P}} \mathscr{E}(P) \to \bigcup_{P\in\mathscr{P}} \mathscr{E}(P)$$

such that:

- $f(e)$ differs from e by some translation τ_e for every $e \in \mathscr{E}(P)$.
- $n_{f(e)} = -n_e$.

Then a *translation surface* is defined as the quotient space

$$M := \coprod_{P\in\mathscr{P}} P \Big/ e \sim f(e)$$

and with all vertices of degree greater than 2 removed.

Example 5.2. Figure 5.1 gives an example of a translation surface, obtained by identifying those edges of the same color of three rectangles. The resulting surface M is not compact as it has a single puncture, which corresponds to an end of the surface. Removing this point gives a compact surface of finite genus.

We may compute its genus using Euler characteristic

$$\chi = v - e + f = 2 - 2g$$

where v (e, f) is the number of vertices (edges, faces) and g is the genus of the surface. There is a unique vertice: notice that in the figure, the point A is identified with C' after gluing together the red edges, and then with C (through the blue edges) $\longrightarrow A'$ (green) $\longrightarrow D'$ (yellow) $\longrightarrow B$ (red) $\longrightarrow B'$ (blue) $\longrightarrow D$ (green).

Clearly there are four edges (taking also the dashed edges into account) and three faces. So

$$\chi = 1 - 6 + 3 = 2 - 2g \quad \Longrightarrow \quad g = 2.$$

The point we have removed is called a *cone-angle singularity*, as it does not have an Euclidean neighbourhood: wrapping around this vertice gives an angle of 6π. A formal definition is that there exists $k \geq 2$ such that the angle around this point is $2\pi k$.

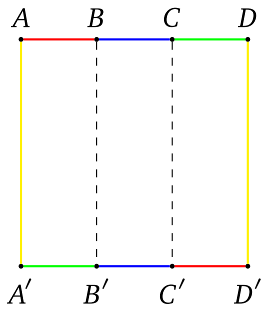


Figure 5.1: An example of a translation surface

Definition 5.3. A translation surface S has *finite type* if S has finite area, and is isomorphic to a compact Riemann surface after a finite number of points removed. S is said to of *infinite type* if it does not have finite type.

Now we provide an "intrinsic" definition of translation surfaces.

Definition 5.4 (Second definition, by translation atlas)**.** A *translation atlas* on a topological surface S is an atlas $\{(U_i, \varphi_i : U_i \to \mathbb{C})\}$ of S such that the transition maps $\varphi_i \circ \varphi_j^{-1}$ are translations on their domains.

A *translation surface* is a topological surface together with a translation atlas thereon.

Example 5.5 (Infinite staircases). Figure 5.2 is known as the *infinite staircases*, which is obtained by identifying the edges that are facing each other. There are four cone-angle singularies A, B, C, D, each of which having infinite degree. It is not hard to see that the surface has infinite genus.

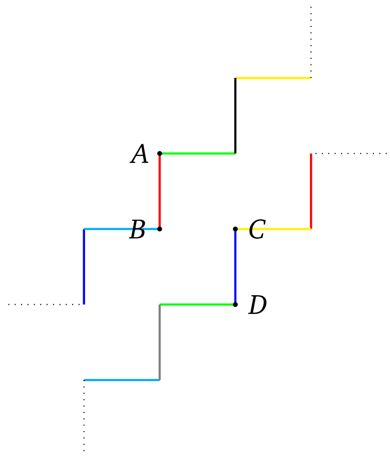


Figure 5.2: The infinite staircases

Example 5.6 (Chamanara surface). Figure 5.3 displays the baker's (or Chamanara) surface introduced in [7]. Consider a square with sides of length 1. Divide each side consecutively into segments, with the n-th segment of length $\frac{1}{2^n}$. Those segments are viewed as edges of a polygon, and those of the same length which lie on opposite sides are identified. This gives a translation surface, which has finite area because it comes from a square of area 1; and infinite genus. It has only one singularity, but a quite singular one, not even an infinite cone-angle singularity because there is no (infinite) covering with Euclidean disks. Such a singularity is called *wild*.

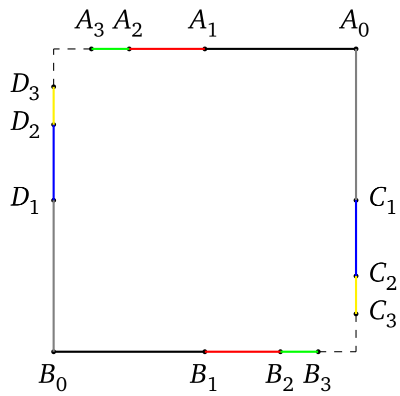


Figure 5.3: The baker's (or Chamanara) surface

5.2 Translation flows

Let $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. For each $z \in \mathbb{C}$ we may define a *parallel flow* $(F^t_{\mathbb{C},\vartheta})_{t\in[0,\infty)}$ at z:

$$F^t_{\mathbb{C},\vartheta} : \mathbb{C} \to \mathbb{C} \qquad z \mapsto z + t\exp(i\vartheta).$$

The flows generate the vector field

$$X_{\mathbb{C},\vartheta} = \left.\frac{\partial F^t_{\mathbb{C},\vartheta}}{\partial t}\right|_{t=0}(z).$$

Let S be a translation surface with a translation atlas $\{(U_i, \varphi_i : U_i \to \mathbb{C})\}$. Pulling back the vector field $X_{\mathbb{C},\vartheta}$ along charts gives vector fields $X_{S,\vartheta}$ on each chart domain. These vector fields can be glued together because the charts differ only by translations. Thus $X_{S,\vartheta}$ is a vector field defined on the whole of S.

Definition 5.7. A *translation flow* is the collection of maximal integral curves of $X_{S,\vartheta}$.

5.3 Bratteli diagrams

5.3.1 Motivation

The identification of segments on opposite sides of the Chamanara surface (Example 5.6) is also called the *Kakutani–von Neumann map*, which gives an infinite interval exchange transformation (see [18]) on $[0,1)$. Namely, the dynamics is generated by the translations

$$[\tfrac{1}{2},1) \to [0,\tfrac{1}{2}), \quad [\tfrac{1}{4},\tfrac{1}{2}) \to [\tfrac{1}{2},\tfrac{3}{4}), \quad [\tfrac{1}{8},\tfrac{1}{4}) \to [\tfrac{3}{4},\tfrac{7}{8}), \quad \ldots$$

Adding the limit points yields a dynamical system on the Cantor set. The new dynamical system is not conjugate to the interval exchange transformation, but since the limit points are countable, we are still able to equip this Cantor dynamical system with an invariant measure. The Kakutani–von Neumann map is a simple example of an infinite interval exchange, and more complicated cases might be found in [4] and the references therein.

A Cantor dynamical system gives an ordered Bratteli diagram using the *Kakutani–Rokhlin partitions* (see [3, Chapter 5]). The resulting ordered Bratteli diagrams are not unique, yet equivalent in a suitable sense. Conversely, starting from an ordered Bratteli diagram one can construct a Cantor dynamical system which is inverse to the previous construction. This is a well-known result from [10]. That says, we have a beautiful dictionary

$$\text{“Dynamics on the Cantor set”} \quad \overset{\sim}{\longleftrightarrow} \quad \text{“Dynamics of an ordered Bratteli diagram”.}$$

The simplest case for the Cantor dynamical system generated by the Kakutani–von Neumann map on the unit interval is given by the *adding machine* (Figure 5.4). The dynamics on a Cantor set can be described by its Bratteli diagram, and the transformation is given by the *Vershik map*, which roughly speaking sends every infinite path to its successor, and the maximal path to the minimal path.

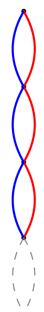


Figure 5.4: The adding machine, which is an ordered Bratteli diagram

5.3.2 A glimpse of ordered bi-infinite Bratteli diagrams

A Bratteli diagram, roughly speaking, is a directed graph[3] whose set of vertices is $\mathbb{N}$-graded (as a set) by finite subsets, and an edge increases the degree by 1. A bi-infinite Bratteli diagram is similar but its set of vertices is $\mathbb{Z}$-graded. More precisely, we have:

Definition 5.8 ([20, Definition 2.1]). A *Bratteli diagram* is a quadruple $\mathscr{B} = (V, E, r, s)$, where V and E are two sets and $r, s \colon E \rightrightarrows V$ are *surjective* maps between them, such that V and E are disjoint unions of non-empty *finite* sets:

- $V = \coprod_{n \in \mathbb{N}_{\geq 0}} V_n$, and $V_0 = \{v_0\}$ is a singleton.
- $E = \coprod_{n \in \mathbb{N}_{\geq 1}} E_n$, such that $r(E_n) = V_n$ and $s(E_n) = V_{n-1}$.

An element in V is called a *vertice* and an element in E is called an *edge*. The maps r and s are called the *range* and *source* maps.

Definition 5.9 ([20, Definition 2.2]). A *bi-infinite Bratteli diagram* is a quadruple $\mathscr{B} = (V, E, r, s)$, defined similarly as in Definition 5.8, but replacing the conditions for V and E by:

- $V = \coprod_{n \in \mathbb{Z}} V_n$.
- $E = \coprod_{n \in \mathbb{Z}} E_n$, such that $r(E_n) = V_n$ and $s(E_n) = V_{n-1}$.

Figure 5.5 is an example of a Bratteli diagram, which is a *stationary* one as it has repeated pattern.

Definition 5.10 ([20, Definition 3.1]). Let $\mathscr{B}$ be a Bratteli diagram or a bi-infinite Bratteli diagram. Denote by $X_{\mathscr{B}}$ the set of *infinite paths* in $\mathscr{B}$. That is, an infinite word

$$x = (\dots, x_{i+1}, x_i, \dots, x_2, x_1) \quad \in \quad \prod_{i \in \mathbb{N}_{\geq 1}} E_i,$$

or a bi-infinite word

$$x = (\dots, x_{i+1}, x_i, \dots, x_2, x_1, \dots) \quad \in \quad \prod_{i \in \mathbb{Z}} E_i,$$

such that $s(x_{i+1}) = r(x_i)$ for all i.

We equip $X_{\mathscr{B}}$ with the Tychonoff topology on $\prod_{i \in \mathbb{N}_{\geq 1}} E_i$ or $\prod_{i \in \mathbb{Z}} E_i$.

[3]For more on graph theory, see Yufan's talk on graph C*-algebras in [17], or Adam's talk on topological graphs in [16].

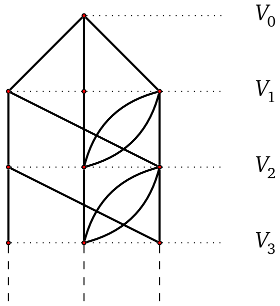


Figure 5.5: An example of a Bratteli diagram

Let $\mathscr{B}$ be a Bratteli diagram or a bi-infininite Bratteli diagram. We are going to define a partial order on $X_{\mathscr{B}}$. This requires an order, such that all edges outgoing a fixed vertice are comparable. This induces a partial order on $X_{\mathscr{B}}$, in which two paths are comparable if they are *tail equivalent*.

Definition 5.11 ([20, Definition 3.5]). Let $\mathscr{B}$ be a Bratteli diagram. Let $x = (x_i)_{i\in\mathbb{N}_{\geq 1}}$ and $y = (y_i)_{i\in\mathbb{N}_{\geq 1}}$ be infinite paths of $\mathscr{B}$. We say they are *tail equivalent*, if there exists some $n \in \mathbb{N}_{\geq 1}$, such that $e_i = f_i$ for all $i \geq n$.

Similar, let $\mathscr{B}$ be a bi-infinite Bratteli diagram. Let $x = (x_i)_{i\in\mathbb{Z}}$ and $y = (y_i)_{i\in\mathbb{Z}}$ be infinite paths of $\mathscr{B}$. We say they are *left* (or *right*) *tail equivalent*, if there exists some $n \in \mathbb{Z}$, such that $e_i = f_i$ for all $i \geq n$ (or $i \leq n$).

Below we will define an order on the path space of a bi-infinite Bratteli diagram. The definition can be easily translated to Bratteli diagrams as well.

Definition 5.12 ([20, Definition 2.10]). An *ordered* bi-infinite Bratteli diagram is a bi-infinite Bratteli diagram $\mathscr{B} = (V, E, r, s)$ together with partial orders $\leq_s$, $\leq_r$ on E, such that for every pair of edges $e, e' \in E$:

- e and e' are $\leq_s$ comparable, if $s(e) = s(e')$.
- e and e' are $\leq_r$ comparable, if $r(e) = r(e')$.

That means that $\leq_s$ (resp. $\leq_r$) is a partial order which restricts to a linear order on $s^{-1}(v)$ (resp. $r^{-1}(v)$) for every $v \in V$.

We write $e <_s e'$ (resp. $e <_r e'$) if $e \leq_s e'$ (resp. $e \leq_r e'$) and $e \neq e'$.

Definition 5.13 ([20, Lemma 4.4]). Let $\mathscr{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram with partial orders $\leq_s$ and $\leq_r$. We define partial orders $\leq_s$ and $\leq_r$ on $X_{\mathscr{B}}$ as follows. Given infinite paths $x = (x_i)_i$ and $y = (y_i)_i$, we say:

- $x \leq_r y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \geq n$ and $x_n \leq_r y_n$.
- $x \leq_s y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \leq n$ and $x_n \leq_s y_n$.

Notice that this means that x and y are comparable in $\leq_r$ (resp. $\leq_s$) iff they are left (resp. right) tail equivalent.

October 31, 2023

The surface associated with a bi-infinite Bratteli diagram

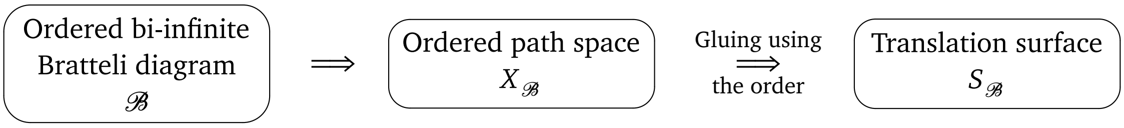
Speaker: Dimitris Gerontogiannis (Leiden University)

In this lecture, we will work towards the construction of a translation surface from an ordered, bi-infinite Bratteli diagram $\mathscr{B}$. More details and properties will be provided in the next talk of Malte.

Along the way, we will also construct measures on the "leaves" of the path space $X_{\mathscr{B}}$, which are later utilised to build a Haar system on a topological groupoid coming from $\mathscr{B}$.

The idea of [20] is very similar with those parallel works on Smale spaces (c.f. [19] and references therein). Starting from a Smale space (X,φ), where X is a compact metric space and $\varphi : X \to X$ is a homeomorphism, one may "discretise" its dynamics using the *Markov partitions* to obtain a symbolic dynamical system (Σ,σ), which assigns to each $x \in X$ a binary code. This gives a Cantor dynamical system. From a sequence in (Σ,σ) viewed as the "binary expansion" of a number, we have natural equivalence relations thereamong. This generates a dynamical system $(\Sigma/\sim, \sigma/\sim)$ on the interval.

Putnam and Treviño used a similar idea to build their translation surfaces, displayed by the following diagram:



A cliffhanger is that, with this order one is able to find the "singularities" of the surface!

Throughout this lecture, we will be extentively using the ordered bi-infinite Bratteli diagram in Figure 6.1 to display examples. For each $n \in \mathbb{Z}$, V_n consists of a unique vertex v_n; and for each n, there are two direct paths from v_n to v_{n+1} in E_n labelled by 1 and 0 satisfying $1 \geq 0$. Note that we have "transposed" the diagram as opposed to the standard convention. This is helpful as we are concerning about bi-infinite Bratteli diagrams.

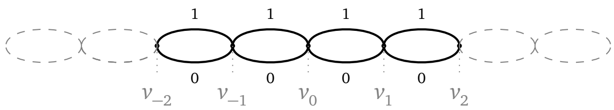


Figure 6.1: A bi-finite Bratteli diagram, used to display examples from this lecture

6.1 Topology and measures on the path space

From now on, we will assume that $\mathscr{B} = (V,E,r,s)$ is a bi-infinite Bratteli diagram.

Definition 6.1. A *state* on $\mathscr{B}$ is a pair of functions

$$(\nu_r, \nu_s), \qquad \nu_s, \nu_s : V \rightrightarrows [0,+\infty)$$

such that

$$\nu_r(v) = \sum_{e \in r^{-1}(v)} \nu_r(s(e)), \qquad \nu_s(v) = \sum_{e \in r^{-1}(v)} \nu_s(r(e))$$

hold for all $v \in V$.

We say a state ν is *normalised*, if $\sum_{v \in V_0} \nu_r(v)\nu_s(v) = 1$.

We say a state is *faithful*, if $\nu_r(v) > 0$ and $\nu_s(v) > 0$ for all $v \in V$.

Example 6.2. A normalised state on the diagram of Figure 6.1 is given by

$$\nu_r(v) = 2^n, \qquad \nu_s(v) = 2^{-n}$$

for all $n \in \mathbb{Z}$ and $v \in V_n$.

Easily one can prove the following "shift invariance" property of states:

Proposition 6.3. *Let (ν_r, ν_s) be a state on $\mathscr{B}$. Then for all $n \in \mathbb{Z}$:*

$$\sum_{v \in V_n} \nu_r(v)\nu_s(v) = \sum_{v \in V_0} \nu_r(v)\nu_s(v).$$

Proposition 6.4. *Every bi-infinite Bratteli diagram $\mathscr{B}$ possesses a state. If $\mathscr{B}$ is* simple *(Definiton 6.6), then every state is faithful.*

Recall that $X_{\mathscr{B}}$ denotes the (bi-infinite) path space of $\mathscr{B}$, whose elements are bi-infinite words

$$x = (\dots, x_{i+1}, x_i, \dots, x_2, x_1, \dots) \quad \in \quad \prod_{i\in\mathbb{Z}} E_i,$$

such that $s(x_{i+1}) = r(x_i)$ for all i.

Definition 6.5. Let $v \in V_n$, define X_v^+ as the space of uni-infinite paths starting at v, whose elements consists of uni-infinite words

$$x = (\dots, x_{i+1}, x_i, \dots, x_{n+2}, x_{n+1}) \quad \in \quad \prod_{i\geq n+1} E_i$$

such that $s(x) := s(x_{n+1}) = v$.

Similarly we define X_v^- as the space of uni-infinite paths ending at v.

Definition 6.6 ([20, Definition 2.4, Lemma 3.3])**.** We say $\mathscr{B}$ is *simple*[4], if for every $m \in \mathbb{Z}$, there are integers l and n with $l < m < n$, such that:

1. There is a path from every vertex of V_l to every vertex of V_m.
2. There is a path from every vertex of V_m to every vertex of V_n.

We say $\mathscr{B}$ is *strongly simple*, if $\mathscr{B}$ is simple and $X_v^{\pm}$ are infinite for all $v \in V$. Due to the simplicity of $\mathscr{B}$, this is equivalent to "…for some $v \in V$".

Recall that $X_{\mathscr{B}}$ is equipped with the Tychonoff topology (product topology) of $\prod_{i\in\mathbb{Z}} E_i$. In particular, we have:

Proposition 6.7. *Equip $X_{\mathscr{B}} \subseteq \prod_{i\in\mathbb{Z}} E_i$ with the Tychonoff topology. Then:*

1. *$X_{\mathscr{B}} \subseteq \prod_{i\in\mathbb{Z}} E_i$ is closed, hence a compact Hausdorff space.*
2. *$X_{\mathscr{B}}$ is* totally disconnected*, whose clopen basis consists of subsets of the form*

$$X_{s(p)}^- p X_{r(p)}^+, \qquad p \text{ is a finite path.}$$

   Those subsets are called cylinder sets *of $X_{\mathscr{B}}$.*

3. *$X_{\mathscr{B}}$ carries an* ultrametric[5]*, given by*

$$d(x,y) := \inf_{n\geq 0} \left\{ 2^{-n} \,\middle|\, x_i = y_i \text{ for all } 1-n \leq i \leq n \right\}.$$

4. *If $\mathscr{B}$ is strongly simple, then $X_{\mathscr{B}}$ is Cantor (i.e. does not have isolated points).*

A state on $\mathscr{B}$ generates a (probability) measure on $X_{\mathscr{B}}$:

Proposition 6.8 ([20, Lemma 3.8])**.** *Let (ν_r, ν_s) be a state on $\mathscr{B}$. Then there is a unique probability measure $\nu_r \times \nu_s$ on $\mathscr{B}$ such that*

$$\nu_r \times \nu_s (X_{s(p)}^- p X_{r(p)}^+) = \nu_r(s(p))\,\nu_s(r(p)).$$

If (ν_r, ν_s) is faithful, then $\nu_r \times \nu_s$ has full support.
If $\mathscr{B}$ is strongly simple, then $\nu_r \times \nu_s$ has no atoms.

[4] There was a mistake in a previous version. I thank Malte Leimbach for pointing it out.

[5] An ultrametric is a metric d such that $d(x,y) \leq \max\{d(y,z), d(x,z)\}$ for all z.

6.2 Topology and measures on the leaves

Now we define the leaves, roughly as the tail equivalence classes of $X_{\mathscr{B}}$. This is, however, not precise, because those are not connected.

Definition 6.9. Let $x \in X_{\mathscr{B}}$ and $N \in \mathbb{Z}$. Define

$$T_N^+(x) := \{y \in X_{\mathscr{B}} \mid y_n = x_n \text{ for all } n > N\},$$
$$T_N^-(x) := \{y \in X_{\mathscr{B}} \mid y_n = x_n \text{ for all } n \leq N\}.$$

The set of right (resp. left) tail equivalence classes is

$$T^+(x) := \bigcup_{N \in \mathbb{Z}} T_N^+(x),$$
$$\text{resp.} \qquad T^-(x) := \bigcup_{N \in \mathbb{Z}} T_N^-(x).$$

Note that $T^{\pm}(x)$ are subsets of $X_{\mathscr{B}}$. However, the density of orbits (equivalence classes) in $X_{\mathscr{B}}$ makes their subspace topologies trivial and hence not useful for our purpose. The correct topology is the *inductive limit topology*: We equip $T_N^{\pm}(x)$ with the relative topology on $X_{\mathscr{B}}$ and equip $T^{\pm}(x)$ with their inductive limit topology. Recall that this means $A \subseteq T^{\pm}(x)$ is open iff $A \cap T_N^{\pm}(x)$ is open for all N.

This is similar with equipping $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$ with the inductive limit topology coming from the natural topologies on intervals. In that case we recovers the standard topology on $\mathbb{R}$.

The following proposition is the one-side version of Proposition 6.8:

Proposition 6.10 ([20, Proposition 3.9])**.** *Let $x = (\ldots, x_{i+1}, x_i, \ldots) \in X_{\mathscr{B}}$. Then there exists a measure ν_r^x on $T^+(x)$ such that*

$$\nu_r^x(X_{s(p)}^- p x_{(n,+\infty)}) = \nu_r(s(p_{m+1}))$$

for all $p \in E_{m,n}$ satisfying $r(p) = r(x_n)$, where

$$x_{(n,+\infty)} := (\ldots, x_{n+2}, x_{n+1}) \in \prod_{i>n} E_i,$$
$$E_{m,n} := \prod_{m<i\leq n} E_i = \{\text{Finite paths from } V_m \text{ to } V_n\}.$$

Similarly, there exists a measure ν_s^x on $T^+(x)$ such that

$$\nu_s^x(x_{(-\infty,m]} p X_{r(p)}^+) = \nu_s(r(p_n))$$

for all $p \in E_{m,n}$ satisfying $s(p) = s(x_m)$.

If (ν_r, ν_s) is faithful, then ν_r^x and ν_s^x have full support. If $\mathscr{B}$ is strongly simple, then ν_r^x and ν_s^x have no atoms.

6.3 Orders on the path space

6.3.1 On finite paths

Recall that (Definition 5.12) an *ordered* bi-infinite Bratteli diagram is a bi-infinite Bratteli diagram $\mathscr{B} = (V, E, r, s)$ together with partial orders $\leq_s$, $\leq_r$ on E, such that for every pair of edges $e, e' \in E$:

- e and e' are $\leq_s$ comparable, if $s(e) = s(e')$.
- e and e' are $\leq_r$ comparable, if $r(e) = r(e')$.

This extends to a partial order on all *finite* paths using the *lexicographic order*.

Example 6.11. Consider the two finite paths $\wr$ and $\wr$ in Figure 6.2. They have the same starting and ending points and hence comparable in both $\leq_r$ and $\leq_s$ using the lexicographic order, which can be viewed as comparing two decimals in the binary expansion. In the $\leq_s$-comparison of two paths we look for the first different edge in those paths from the *left*, then read "from left to right". Then the comparison reads

$$0011 \leq_s 1001.$$

The $\leq_r$-comparison is from the opposite direction, namely compare the first different edge from the *right*. Thus

$$0011 \geq_r 1001.$$

(a) $\wr$=0011 (b) $\wr$=1001

Figure 6.2: "Binary expansions" of two finite paths $\wr$ and $\wr$

Definition 6.12. Let X be a linearly ordered set and $x, y \in X$. We say y is the *successor* of x and x is the *predecessor* of y if $x < y$ and there is no z such that $x < z < y$.

For every edge $e \in E$, since $s^{-1}(s(e))$ is linearly ordered in $\leq_s$ and $r^{-1}(r(e))$ is linearly ordered in $\leq_r$, we are able to talk about the *successor* or *predecessor* of e in $\leq_r$ or $\leq_s$, providing those exist. These notions can be extended to finite paths as well.

Example 6.13. In Figure 6.3, $\wr$ has $\leq_s$-successor $\wr$, and $\wr$ has $\leq_s$-predecessor $\wr$.

(a) $\wr$=0000 (b) $\wr$=0001

Figure 6.3: $\wr$ is the $\leq_s$-successor of $\wr$

6.3.2 On infinite paths

Definition and Lemma 6.14 ([20, Proposition 4.1])**.** $\mathscr{B}$ contains an infinite path, such that every edge of it is s-maximal (resp. r-maximal, resp. s-minimal, resp. r-minimal). We denote the collection of such paths by $X_{\mathscr{B}}^{s\text{-max}}$ (resp. $X_{\mathscr{B}}^{r\text{-max}}$, resp. $X_{\mathscr{B}}^{s\text{-min}}$, resp. $X_{\mathscr{B}}^{r\text{-min}}$). Each of them is closed in $X_{\mathscr{B}}$.

If there exists some K such that $\#V_n \leq K$ for all $n \in \mathbb{Z}$. Then the cardinal of each of these sets is at most K.

We define

$$X_{\mathscr{B}}^{\text{ext}} := X_{\mathscr{B}}^{s\text{-max}} \cup X_{\mathscr{B}}^{r\text{-max}} \cup X_{\mathscr{B}}^{s\text{-min}} \cup X_{\mathscr{B}}^{r\text{-min}}.$$

Recall the partial orders $\leq_s$ and $\leq_r$ on $X_{\mathscr{B}}$ (Definition 5.13): given infinite paths $x = (x_i)_i$ and $y = (y_i)_i$, we say:

- $x \leq_r y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \geq n$ and $x_n \leq_r y_n$.
- $x \leq_s y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \leq n$ and $x_n \leq_s y_n$.

So x and y are comparable in $\leq_r$ (resp. $\leq_s$) iff they are left (resp. right) tail equivalent. In particular, this means that $T^-(x)$ (resp. $T^+(x)$) is linearly $\leq_s$-ordered (resp. $\leq_r$-ordered). Thus for infinite paths that are in the same tail equivalence class, we are able to compare them and speak about their successors and predecessors as well providing they exist.

6.4 Defining the surface

From now on, we assume that $\mathscr{B}$ is strongly simple.

Definition 6.15. Define the *source boundary* $\partial_s X_{\mathscr{B}}$ of $X_{\mathscr{B}}$ as

$$\partial_s X_{\mathscr{B}} := \{x \in X_{\mathscr{B}} \mid x \text{ has either an } s\text{-successor or an } s\text{-predecessor}\}.$$

Remark 6.16. We claim (without proving here) that an infinite path cannot have both an s-successor or an s-predecessor. A good example is to consider the decimal expansion of reals, which can be viewed as a quotient of a Cantor denary system: the decimal

$$0.99999\cdots$$

has a successor $1.0000\cdots$. They are not yet equivalent in the Cantor set, but "identified" and hence becoming representatives of the same *real number*. However, $0.999\cdots$ does not have a predecessor.

Definition 6.17. Define

$$\Delta_s : \partial_s X_{\mathscr{B}} \to \partial_s X_{\mathscr{B}}$$

by sending x to its s-successor or s-predecessor. The previous remark guarantees that this map is indeed well-defined.

Clearly we have $\Delta_s^2 = \mathrm{id}$. Similarly we define $\partial_r X_{\mathscr{B}}$ and $\Delta_r X_{\mathscr{B}}$. Notice that $\Delta_r(x)$ (resp. $\Delta_s(x)$) and x are right (resp. left) tail equivalent.

Definition and Lemma 6.18. Define the boundary of $X_{\mathscr{B}}$ as

$$\partial X_{\mathscr{B}} := \partial_s X_{\mathscr{B}} \cap \partial_r X_{\mathscr{B}}.$$

Then Δ_s and Δ_r leave $\partial X_{\mathscr{B}}$ invariant. The set of *singular points* of $X_{\mathscr{B}}$ is defined as

$$\Sigma_{\mathscr{B}} := \{x \in \partial X_{\mathscr{B}} \mid \Delta_r \circ \Delta_s(x) \neq \Delta_s \circ \Delta_r(x)\}.$$

Why is such a point "singular"? Well, as we shall anticipate, gluing the path space $X_{\mathscr{B}}$ using the order yields a flat surface. But the condition $\Delta_r \circ \Delta_s(x) \neq \Delta_s \circ \Delta_r(x)$ says that the surface is not "flat" around x.

Example 6.19. Figure 6.4 displays an example of a singular point $\wr$. In the "binary expansion" it can be written as $\cdots 00100 \cdots$, which belongs to the boundary of $X_{\mathscr{B}}$. But:

$$\cdots 00100 \cdots \xrightarrow{\Delta_s} \cdots 00011 \cdots \xrightarrow{\Delta_r} \cdots 11101 \cdots;$$
$$\cdots 00100 \cdots \xrightarrow{\Delta_r} \cdots 11000 \cdots \xrightarrow{\Delta_s} \cdots 10111 \cdots.$$

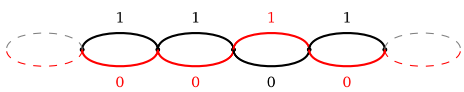


Figure 6.4: Example of a singular point $\wr = \cdots 00100 \cdots$

Definition and Lemma 6.20. Define

$$Y_{\mathscr{B}} := X_{\mathscr{B}} \setminus (X_{\mathscr{B}}^{\mathrm{ext}} \cup \Sigma_{\mathscr{B}}).$$

Both $X_{\mathscr{B}}^{\mathrm{ext}}$ and $\Sigma_{\mathscr{B}}$ are countable, closed subsets of $X_{\mathscr{B}}$. So $Y_{\mathscr{B}}$ is open.

The surface $S_{\mathscr{B}}$ is

$$S_{\mathscr{B}} := Y_{\mathscr{B}} / \sim$$

with the equivalence relation $\sim$ generated by:

$$y \sim \Delta_s(y), \qquad \text{if } y \in \partial_s X_{\mathscr{B}} \cap Y_{\mathscr{B}};$$
$$y \sim \Delta_r(y), \qquad \text{if } y \in \partial_r X_{\mathscr{B}} \cap Y_{\mathscr{B}}.$$

November 14, 2023

Translation surfaces, groupoids and C*-algebras

Speaker: Malte Leimbach (Radboud University Nijmegen)

7.1 Translation surfaces

Let $\mathscr{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram, with orders $\leq_s$ and $\leq_r$.

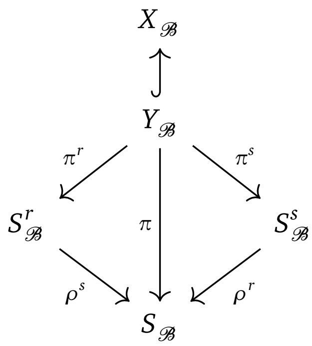
Recall from Dimitris's talk that we have removed singular and boundary points of $X_{\mathscr{B}}$ to get $Y_{\mathscr{B}}$. The translation surface $S_{\mathscr{B}}$ is obtained by gluing certain pairs of points in $Y_{\mathscr{B}}$, using the orders of $\mathscr{B}$ and $X_{\mathscr{B}}$, see Definition and Lemma 6.20. We explain this construction in more detail.

We introduce first more symbols and notations. For our convenience, a complete list of symbols used in [20] is provided in the notes.

Definition 7.1. Define

$$S^s_{\mathscr{B}} := Y_{\mathscr{B}} \Big/ x \sim \Delta_s(x),\ x \in Y_{\mathscr{B}} \cap \partial_s X_{\mathscr{B}};$$
$$S^r_{\mathscr{B}} := Y_{\mathscr{B}} \Big/ x \sim \Delta_r(x),\ x \in Y_{\mathscr{B}} \cap \partial_r X_{\mathscr{B}}.$$

Then we have an commuting diagram consisting of $X_{\mathscr{B}}$, $Y_{\mathscr{B}}$, $S^s_{\mathscr{B}}$, $S^r_{\mathscr{B}}$, $S_{\mathscr{B}}$ with obvious quotient maps between them:



To finalise the construction of a translation surface $S_{\mathscr{B}}$, we must construct translation atlas for it.

Definition 7.2 ([20, Definition 6.1])**.** Denote by $E^Y_{m,n}$ the collection of finite paths $p \in E_{m,n}$ with:

- p is neither s-maximal, s-minimal, r-maximal nor r-minimal.
- $X^-_{s(p)} p X^+_{r(p)} \subseteq Y_{\mathscr{B}}$.

Definition 7.3 ([20, Definition 6.9])**.** Denote by $E^{s/r}_{m,n}$ the set of quadruples

$$p = (p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}), \qquad p_{i,j} \in E^Y_{m,n},$$

such that:

- $p_{i,j+1}$ is an s-successor of $p_{i,j}$.
- $p_{i+1,j}$ is an r-successor of $p_{i,j}$.

Namely, we have the following diagram of successors:

$$\begin{array}{ccc} p_{1,1} & \xrightarrow{s\text{-succ.}} & p_{1,2} \\ {\scriptstyle r\text{-succ.}}\downarrow & & \downarrow{\scriptstyle r\text{-succ.}} \\ p_{2,1} & \xrightarrow[s\text{-succ.}]{} & p_{2,2} \end{array}$$

Definition 7.4 ([20, Definition 6.9, Theorem 6.13]). Define

$$V_{i,j}(p) := \left(X^{-}_{s(p_{i,j})} \setminus X^{r\text{-}m_i}_{s(p_{i,j})}\right) p_{i,j} \left(X^{+}_{r(p_{i,j})} \setminus X^{s\text{-}m_j}_{r(p_{i,j})}\right)$$

where $i, j \in \{1, 2\}$, $m_1 = \min$, $m_2 = \max$.

Define

$$V(p) := \bigcup_{i,j} V_{i,j}(p) \subseteq Y_{\mathscr{B}}, \qquad Y(p) := \pi(V(p)) \subseteq S_{\mathscr{B}}.$$

Example 7.5. Figure 7.1 gives an example of $V_{1,1}$ where $p_{1,1}$ is the path $\wr$ of length 1 connecting v_0 and v_1 labelled by 1. Then

$$V_{1,1}(p) := \left(X^{-}_{v_0} \setminus X^{r\text{-}\min}_{v_0}\right) p_{1,1} \left(X^{+}_{v_1} \setminus X^{s\text{-}\min}_{v_1}\right).$$

It consists of (bi-infinite) paths x in $\mathscr{B}$, satisfying:

1. x contains the path $p_{1,1} = \wr$ as a segment.
2. x does *not* contain the minimal path ending at v_0: namely, the dashed path $\wr = \cdots 000$ which ends at v_0.
3. x does *not* contain the minimal path starting from v_0: namely, the dashed path $\wr = 000\cdots$ which starts from v_0.

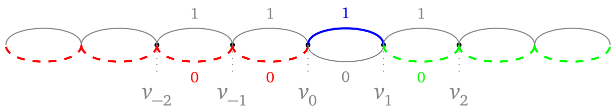


Figure 7.1: Example of $V_{1,1}$

We will see that $V(p)$'s are the chart domains for the translation atlas of $S_{\mathscr{B}}$. To construct the charts, we need the following

Definition and Lemma 7.6 ([20, Proposition 3.7, Lemma 3.8]). Let (ν_s, ν_r) be a normalised state on $\mathscr{B}$ (see Definition 6.1). Denote by $X^{+}_{\mathscr{B}}$ (resp. $X^{-}_{\mathscr{B}}$) the set of right-infinite (resp. left-infinite) paths of $\mathscr{B}$. There there is a unique measure ν_s on $X^{+}_{\mathscr{B}}$ such that

$$\nu_s(pX^{+}_{r(p)}) = \nu_s(r(p))$$

and a unique measure ν_r on $X^{-}_{\mathscr{B}}$ such that

$$\nu_r(X^{-}_{s(p)}p) = \nu_r(s(p))$$

for any finite path p.

This is proved by truncating the bi-infinite Bratteli diagram $\mathscr{B}$ into a Bratteli diagram which is only left or right infinite, then using [20, Proposition 3.7]. Note that the measure in Proposition 6.8 is a product measure of ν_r and ν_s defined here.

Definition 7.7 ([20, Definition 4.7]). Let $v \in V$. Define

$$\varphi^{v}_{s} : X^{+}_{v} \to [0, \nu_s(v)], \qquad x \mapsto \nu_s(\{y \in X^{+}_{v} \mid y \leq_s x\}),$$
$$\varphi^{v}_{r} : X^{-}_{v} \to [0, \nu_r(v)], \qquad x \mapsto \nu_r(\{y \in X^{-}_{v} \mid y \leq_r x\}).$$

7.1.1 The translation atlas

Now we are able to describe the translation atlas for $S_{\mathscr{B}}$. We assume for now the following **standing assumptions**:

- $\mathscr{B}$ has *finite rank*, that is, V_n is uniformly bounded.
- $\mathscr{B}$ is *strongly simple*, that is, $\mathscr{B}$ is simple and $\#X_v^{\pm} = \infty$ for every $v \in V$. (See Definition 6.6).
- $X_{\mathscr{B}}^{\mathrm{ext}} \cap \partial_r X_{\mathscr{B}} = \emptyset$ and $X_{\mathscr{B}}^{\mathrm{ext}} \cap \partial_s X_{\mathscr{B}} = \emptyset$.

Definition and Lemma 7.8 ([20, Definition 6.9, Theorem 6.13]). Let $p \in E_{m,n}^{r/s}$. Define $V(p) \subseteq Y_{\mathscr{B}}$ and $Y(p) = \pi(V(p)) \subseteq S_{\mathscr{B}}$ as in Definition 7.4.

- We define $\psi^p : V(p) \to \mathbb{R}^2$ by

$$\psi^p(x) := \left(\varphi_r^{s(x)}(x_{(-\infty,m]}), \varphi_s^{r(x)}(x_{[n,+\infty)})\right) + \begin{cases} (-\nu_r(s(p_{1,1})), -\nu_s(r(p_{1,1}))) & x \in V_{1,1}(p), \\ (-\nu_r(s(p_{1,1})), 0) & x \in V_{1,2}(p), \\ (0, -\nu_s(r(p_{1,1}))) & x \in V_{2,1}(p), \\ (0,0) & x \in V_{2,2}(p). \end{cases}$$

- There is a unique map $\eta^p : Y(p) \to \mathbb{R}^2$ satisfying $\eta^p = \psi^p \circ \pi$.

Proposition 7.9. *Define ϕ^p, η^p as in Definition and Lemma 7.8. We have the following properties:*

1. *$\bigcup_{n \in \mathbb{N}} \{V(p)\}_{p \in E_{-n,n}^{r/s}}$ is an open over of $Y_{\mathscr{B}}$.*
2. *$V(p)$ is invariant under Δ_r and Δ_s.*
3. *ψ^p is continuous.*
4. *$\psi^p(x) = \psi^p(y)$ iff $\pi(x) = \pi(y)$.*
5. *For any $p \in E_{-m,m}^{r/s}$ and $q \in E_{-n,n}^{r/s}$, there exists $c_{p,q} \in \mathbb{R}^2$ such that*

$$\psi^p(x) = \psi^q(x) - c_{p,q}, \qquad \textit{for all } x \in V(p) \cap V(q).$$

Finally, we are able to describe the translation atlas for $S_{\mathscr{B}}$:

Theorem 7.10 ([20, Theorem 6.13]). *There exists a sequence of natural numbers $\{n_k\}_{k \geq 1} \subseteq \mathbb{N}$, such that*

$$\left\{ \eta^p : Y_p \to \mathbb{R}^2 \;\middle|\; p \in \bigcup_{k \geq 1} E_{-n_k,n_k}^{r/s} \right\}$$

is a translation atlas for $S_{\mathscr{B}}$.

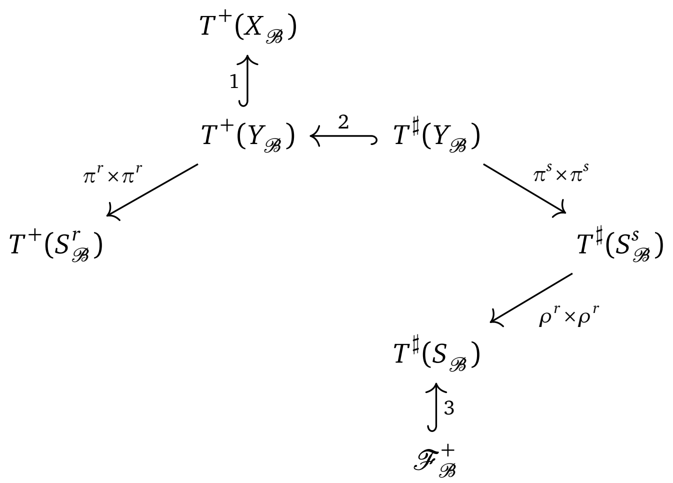
7.2 Groupoids

In the following, we will introduce several groupoids, and construct Haar systems thereon to pass to their (reduced) C*-algebras. Almost all of these groupoids are defined by *equivalence relations*. That is, let X be a topological space, and $\mathscr{R} \subseteq X \times X$ be an equivalence relation on X, equipped with the subspace topology. Then

$$\mathscr{R} \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X$$

is a topological groupoid, with multiplication $(x,y) \cdot (y,z) := (x,z)$ and inverse $(x,y)^{-1} := (y,x)$. As a special case, the pair groupoid is defined by the full equivalence relation $\mathscr{R} := X \times X$.

The groupoids introduced in this section (and in the paper [20]) as well as their relations are described by the diagram below:



(7.1)

By $\mathscr{G} \hookrightarrow \mathscr{H}$ I mean $\mathscr{G}$ is an open subgroupoid of $\mathscr{H}$. The symbol $\sim$ and note "stably equivalent" on these arrows are not about the groupoids themselves, but rather indicate the relation between their C*-algebras, as we will see in the next section.

***$T^+(X_{\mathscr{B}})$* and *$T^+(Y_{\mathscr{B}})$*.** The right tail equivalence relation on $X_{\mathscr{B}}$ generates a groupoid, which we denoted by $T^+(X_{\mathscr{B}})$. It carries the inductive limit topology from $T^+_N(X_{\mathscr{B}})$, the latter defined by "right tail equivalence starting from V_N". With this topology, $T^+(X_{\mathscr{B}})$ is locally compact and Hausdorff.

The Haar system on $T^+(X_{\mathscr{B}})$ is $\{\nu_r^x\}_{x \in X_{\mathscr{B}}}$ as described in Proposition 6.10.

$Y_{\mathscr{B}}$ is obtained by removing countably many points from $X_{\mathscr{B}}$. So there is an open subgroupoid $T^+(Y_{\mathscr{B}})$ defined by restricting $T^+(X_{\mathscr{B}})$ to $Y_{\mathscr{B}} \subseteq X_{\mathscr{B}}$. The groupoid is equipped with the same Haar system with $T^+(X_{\mathscr{B}})$ because a countable set has measure zero.

***$T^+(S^r_{\mathscr{B}})$*. Why not *$T^+(S^s_{\mathscr{B}})$*?** The gluing $\pi^r : Y_{\mathscr{B}} \to S^r_{\mathscr{B}}$ identifies x with $\Delta_r(x)$ in $Y_{\mathscr{B}}$. But note that x and $\Delta_r(x)$ are already right tail equivalent. That says, we "do not obtain very much new information" from this process.

We define

$$T^+(S^r_{\mathscr{B}}) := \pi^r \times \pi^r(T^+(Y_{\mathscr{B}}))$$

and equip it with the quotient topology. $\pi^r \times \pi^r$ is a continuous proper surjection, thus it gives a Haar system $\{\pi^r_* \nu_r^x\}_{x \in S^r_{\mathscr{B}}}$.

The gluing $\pi^s : Y_{\mathscr{B}} \to S^s_{\mathscr{B}}$ is however different, because x and $\Delta_s(x)$ are not right tail equivalent. Thus we cannot define $T^+(X^s_{\mathscr{B}})$ in a similar naïve way. We have to pass to a subgroupoid of $T^+(Y_{\mathscr{B}})$.

***$T^\sharp(Y_{\mathscr{B}})$*.** The correct construction to encode the equivalence generated by Δ_s is given by the following open subgroupoid of $T^+(Y_{\mathscr{B}})$:

$$T^\sharp(Y_{\mathscr{B}}) := \left\{(x,y) \in T^+(Y_{\mathscr{B}}) \mid \text{If } x, y \in \partial_s X_{\mathscr{B}}, \text{ then } (\Delta_s(x), \Delta_s(y)) \in T^+(Y_{\mathscr{B}})\right\}.$$

As an open subgroupoid, restricting the Haar system $\{\nu_r^x\}_{x \in Y_{\mathscr{B}}}$ to it yields a Haar system.

Remark 7.11 ([20, Proposition 7.6]). Note that $T^\sharp(Y_{\mathscr{B}}) = T^+(Y_{\mathscr{B}})$ if

$$\#(\text{tail equivalence classes of } s\text{-min paths}) = \#(\text{tail equivalence classes of } s\text{-max paths}) = 1.$$

***$T^\sharp(S^s_{\mathscr{B}})$*.** Now the construction mimicks that of $T^+(S^r_{\mathscr{B}})$, except that we are working with $T^\sharp(Y_{\mathscr{B}})$. We define

$$T^\sharp(S^s_{\mathscr{B}}) := \pi^s \times \pi^s(T^\sharp(Y_{\mathscr{B}}))$$

and equip it with the quotient topology. $\pi^s \times \pi^s$ is a continuous proper surjection, thus it gives a Haar system $\{\pi^s_* \nu_r^x\}_{x \in S^s_{\mathscr{B}}}$.

$T^\sharp(S_{\mathscr{B}})$. Define

$$T^\sharp(S_{\mathscr{B}}) := \rho^r \times \rho^r(T^\sharp(X^s_{\mathscr{B}})) = \pi \times \pi(T^\sharp(Y_{\mathscr{B}})).$$

with the Haar system $\{\rho^r_* \pi^s_* \nu^x_r\}_{x \in S_{\mathscr{B}}}$

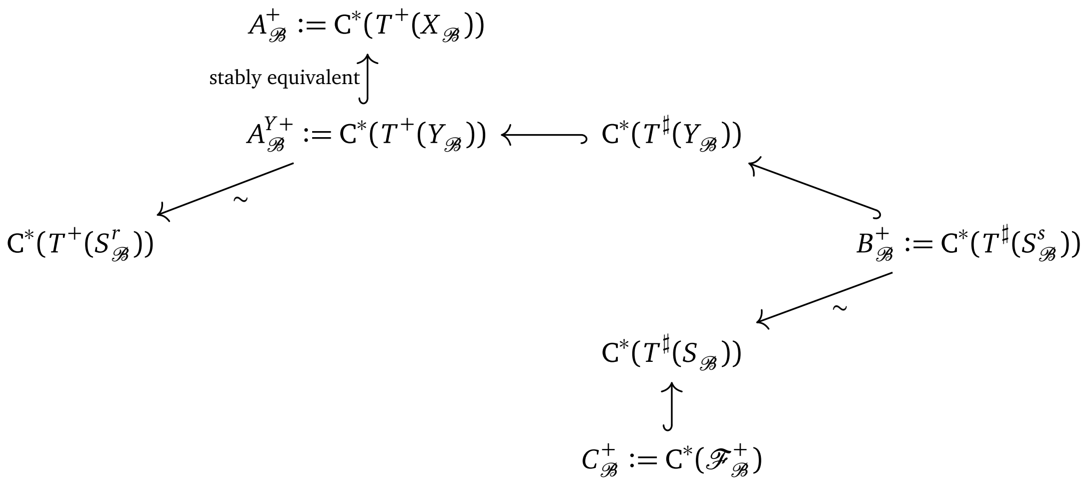
$\mathscr{F}^+_{\mathscr{B}}$. $T^\sharp(S_{\mathscr{B}})$ is almost the foliation groupoid, except that its leaves may not be connected. The last step is to replace the leaves by its connected components:

$$\mathscr{F}^+_{\mathscr{B}} := \{(x,y) \in T^\sharp(S_{\mathscr{B}}) \mid x, y \text{ are in the same connected component}\}.$$

I should leave its topology and Haar system as an exercise.

7.3 C*-algebras

The groupoids from the previous section yield groupoid C*-algebras, whose relations and symbols are shown in the diagram below:



(7.2)

Their K-theory will be discussed thoroughly in the coming talk by Yufan. Many of these C*-algebras have the same K-theory because they are stably isomorphic or even isomorphic. Besides those, we will also define several inductive systems of C*-algebras:

$$(A^+_{m,n}),\ (AC^+_{m,n});\ (A^{Y+}_{m,n}),\ (AC^{Y+}_{m,n});\ (B^+_{m,n});\ (C^+_{m,n}).$$

The first one consists of finite-dimensional C*-algebras, and their union is dense in $A^+_{\mathscr{B}} := \mathrm{C}^*(T^+(X_{\mathscr{B}}))$. This realises $A^+_{\mathscr{B}}$ as an AF-algebra. The other inductive system provides us with useful short exact sequences, allowing us to compute the K-theory of the foliation C*-algebra $C^+_{\mathscr{B}} := \mathrm{C}^*(\mathscr{F}^+_{\mathscr{B}})$.

7.3.1 $A^+_{\mathscr{B}}$, $A^{Y+}_{\mathscr{B}}$, the inductive systems $(A^+_{m,n})$, $(A^{Y+}_{m,n})$, $(AC^+_{m,n})$ and $(AC^{Y+}_{m,n})$

Definition 7.12. We define

$$A^+_{\mathscr{B}} := \mathrm{C}^*(T^+(X_{\mathscr{B}})), \qquad A^{Y+}_{\mathscr{B}} := \mathrm{C}^*(T^+(Y_{\mathscr{B}})).$$

We are going to construct an inductive system $(A^+_{m,n})_{m,n\in\mathbb{Z}}$, each $A_{m,n}$ being a finite-dimensional C*-algebra, and such that

$$A^+_{\mathscr{B}} = \overline{\bigcup_{m,n} A^+_{m,n}}.$$

This realises $A^+_{\mathscr{B}}$ as an AF-algebra.

Before doing that, we claim that the C*-subalgebra $A^{Y+}_{\mathscr{B}}$ is actually stably equivalent to $A^+_{\mathscr{B}}$:

Proposition 7.13. *$A^{Y+}_{\mathscr{B}}$ is a full hereditary* C^**-subalgebra of $A^{+}_{\mathscr{B}}$* [6]*, hence the inclusion $A^{Y+}_{\mathscr{B}} \hookrightarrow A^{+}_{\mathscr{B}}$ induces a stable equivalence of* C^**-algebras (see* [2]*):*

$$A^{Y+}_{\mathscr{B}} \otimes \mathbb{K} \simeq A^{+}_{\mathscr{B}} \otimes \mathbb{K},$$

and an isomorphism in K*-theory.*

This is roughly because we only remove a countable set of points from $X_{\mathscr{B}}$ to obtain $Y_{\mathscr{B}}$. As a consequence, passing from $A^{+}_{\mathscr{B}}$ to $A^{Y+}_{\mathscr{B}}$ is quite minor.

Definition 7.14. Let $p, q \in E_{m,n}$. Define the map $a_{p,q} \colon T^+(X_{\mathscr{B}}) \to \mathbb{R}$ by:

$$a_{p,q}(x,y) := \begin{cases} \nu_r(s(p))^{-1/2}\nu_r(s(q))^{-1/2} & \text{if } x_{(m,n]} = p,\ y_{(m,n]} = q,\ x_{(n,+\infty)} = y_{(n,+\infty)}; \\ 0 & \text{otherwise.} \end{cases}$$

The definition of $a_{p,q}$ reveals them as "propagation kernels" on $T^+(X_{\mathscr{B}})$. This can be seen more clearly in [20, Proposition 8.2].

Definition 7.15. Let $v \in V$. Define

$$A^{+}_{m,n,v} := \operatorname{span}\{a_{p,q} \mid p, q \in E_{m,n},\ r(p) = r(q) = v\},$$

and

$$A^{+}_{m,n} := \bigoplus_{v \in V_n} A^{+}_{m,n,v}.$$

Proposition 7.16 ([20, Proposition 8.3])**.** *Define $A^{+}_{m,n,v}$ and $A^{+}_{m,n}$ as above. Then we have:*

1. *$A^{+}_{m,n,v} \simeq \mathbb{M}_{j(m,n,v)}(\mathbb{C})$, where*

$$j(m,n,v) := \#(\text{paths in } E_{m,n} \text{ with range } v).$$

2. *As a corollary, $A^{+}_{m,n}$ is a finite-dimensional* C^**-algebra.*
3. *$A^{+}_{m-1,n} \subseteq A^{+}_{m,n} \subseteq A^{+}_{m,n+1}$. So $(A^{+}_{m,n})$ is an inductive system.*
4. *$A^{+}_{\mathscr{B}}$ is an AF-algebra given by the inductive system $(A^{+}_{m,n})$, namely*

$$A^{+}_{\mathscr{B}} = \overline{\bigcup_{m,n} A^{+}_{m,n}}.$$

The inductive system $(A^{+}_{m,n})$ for $A^{+}_{\mathscr{B}}$ can be "restricted to" $A^{Y+}_{\mathscr{B}}$ as well:

Definition and Lemma 7.17 ([20, Proposition 8.7])**.** Define

$$A^{Y+}_{m,n,v} := \operatorname{span}\{a_{p,q} \mid p, q \in E^{Y}_{m,n},\ r(p) = r(q) = v\},$$

where $a_{p,q}$ are as in Definition 7.14, and

$$A^{Y+}_{m,n} := \bigoplus_{v \in V_n} A^{Y+}_{m,n,v}.$$

Then $(A^{Y+}_{m,n})$ is an inductive system for $A^{Y+}_{\mathscr{B}}$. Moreover, all conditions of Proposition 7.16 hold after replacing "+" by "$Y+$".

[6] Let A be a C^*-algebra. A C^*-subalgebra $B \subseteq A$ is called *hereditary*, if for every $a \in A$ and $b \in B$, $0 \leq a \leq b$ implies $a \in B$. A hereditary C^*-subalgebra is *full* if it is not contained in any proper closed ideal of A.

Now we construct inductive systems $(AC^{+}_{m,n})$ and $(AC^{Y+}_{m,n})$, which consists of infinite-dimensional C*-algebras. Nevertheless, this inductive system will be useful for the K-theory computation.

Definition 7.18. Let $v \in V$. Define

$$AC^{+}_{m,n,v} := A^{+}_{m,n,v} \otimes \mathrm{C}(X^{+}_{v}),$$

and

$$AC^{+}_{m,n} := \bigoplus_{v \in V_n} AC^{+}_{m,n,v}.$$

Similarly we define $AC^{Y+}_{m,n,v}$ and $AC^{Y+}_{m,n}$ by replacing every "+" with "$Y+$".

Remark 7.19 ([20, Proposition 8.4]). An equivalent definition of $AC^{+}_{m,n,v}$ is by

$$AC^{+}_{m,n,v} := \operatorname{span}\left\{a_{pp',qp'} \mid p,q \in E_{m,n},\, p' \in E_{n,n'},\, r(p) = r(q) = s(p') = v\right\},$$

where pp' and qp' are the concatenation of paths in the usual sense, and $a_{pp',qp'}$ is as in Definition 7.14. The map

$$a_{pp',qp'} \mapsto a_{p,q} \otimes \chi_{p'X^{+}_{r(p')}}$$

extends to an isomorphism $AC^{+}_{m,n,v} \simeq A^{+}_{m,n,v} \otimes \mathrm{C}(X^{+}_{v})$.

7.3.2 $B^{+}_{\mathscr{B}}$, the inductive system $(B^{+}_{-n,n})$

Definition 7.20. Define

$$B^{+}_{\mathscr{B}} := \mathrm{C}^{*}(T^{\sharp}(S^{s}_{\mathscr{B}})).$$

There are inclusions of C*-algebra:

$$B^{+}_{\mathscr{B}} \subseteq \mathrm{C}^{*}(T^{\sharp}(Y_{\mathscr{B}})) \subseteq A^{Y+}_{\mathscr{B}} \subseteq A^{+}_{\mathscr{B}}.$$

We will display $B^{+}_{\mathscr{B}}$ as an inductive limit of C*-algebras $(B^{+}_{-n,n})$ which are not finite-dimensional. Note that a C*-subalgebra of an AF-algebra is typically not AF. So $B^{+}_{\mathscr{B}}$, as a C*-subalgebra of the AF-algebra $A^{Y+}_{\mathscr{B}}$, need not be AF.

Definition 7.21. Define

$$B^{+}_{m,n} := AC^{Y+}_{m,n} \cap B^{+}_{\mathscr{B}}.$$

Then every $B^{+}_{m,n}$ $(m<n)$ is a C*-subalgebra of $B^{+}_{\mathscr{B}}$.

Proposition 7.22 ([20, Theorem 8.9])**.** *Define $B^{+}_{m,n}$ as above. Then:*

1. $B^{+}_{m,n} \subseteq B^{+}_{m-1,n+1}$. *So $(B^{+}_{-n,n})$ is an inductive system.*
2. $B^{+}_{\mathscr{B}}$ *is the inductive limit* C*-*algebra given by the inductive system, that is,*

$$B^{+}_{\mathscr{B}} = \overline{\bigcup_{n} B^{+}_{-n,n}}.$$

The K-theory of $B^{+}_{m,n}$ can be computed by a short exact sequence. For this, we must introduce another inductive system of groupoids.

Definition and Lemma 7.23 ([20, Definition 6.4, Page 43])**.** Denote by $E^{s}_{m,n}$ the set of pairs

$$p = (p_1, p_2), \qquad p_i \in E^{Y}_{m,n},$$

such that p_2 is an s-successor of p_1.

Define $G_{m,n}$ as the collection of pairs of elements in $E^{s}_{m,n}$:

$$G_{m,n} := \{(p,q) \in E^{s}_{m,n} \times E^{s}_{m,n} \mid r(p_1) = r(q_1),\, r(p_2) = r(q_2)\}.$$

Then $G_{m,n}$ is a finite equivalence relation and hence a groupoid.

Proposition 7.24 ([20, Corollary 8.10])**.** *There is a short exact sequence*

$$\bigoplus_{v \in V_n} A^{+}_{m,n,v} \otimes \mathrm{C}_0(0, \nu_s(v)) \rightarrowtail B^{+}_{m,n} \twoheadrightarrow \mathrm{C}^{*}(G_{m,n}).$$

7.3.3 Comparing $A^{Y+}_{\mathscr{B}}$ with $\mathrm{C}^*(T^+(S^r_{\mathscr{B}}))$, $B^+_{\mathscr{B}}$ with $\mathrm{C}^*(T^\sharp(S_{\mathscr{B}}))$

Now we describe the two isomorphisms given in the diagram (7.2).

Theorem 7.25 ([20, Theorem 8.11]). *We have isomorphisms $A^{Y+}_{\mathscr{B}} \simeq \mathrm{C}^*(T^=(S^r_{\mathscr{B}}))$ and $B^+_{\mathscr{B}} \simeq \mathrm{C}^*(T^\sharp(S_{\mathscr{B}}))$. More precisely:*

1. *The map $(\pi^r \times \pi^r)^*\colon \mathrm{C_c}(T^+(S^r_{\mathscr{B}})) \to \mathrm{C_c}(T^+(Y_{\mathscr{B}}))$ induces an isomorphism*

$$A^{Y+}_{\mathscr{B}} \simeq \mathrm{C}^*(T^+(S^r_{\mathscr{B}})).$$

2. *The map $(\rho^s \times \rho^s)^*\colon \mathrm{C_c}(T^\sharp(S_{\mathscr{B}})) \to \mathrm{C_c}(T^\sharp(S^r_{\mathscr{B}}))$ induces an isomorphism*

$$B^+_{\mathscr{B}} \simeq \mathrm{C}^*(T^\sharp(S_{\mathscr{B}})).$$

7.3.4 $C^+_{\mathscr{B}}$, the inductive system $(C^+_{-n,n})$

Definition 7.26. The foliation C^*-algebra is defined as

$$C^+_{\mathscr{B}} := \mathrm{C}^*(\mathscr{F}^+_{\mathscr{B}}).$$

This is a C^*-subalgebra of $\mathrm{C}^*(T^\sharp(S_{\mathscr{B}}))$. We will show that, in a very similar fashion with $B^+_{\mathscr{B}}$: the C^*-algebra $C^+_{\mathscr{B}}$ can be realised as an inductive limit of C^*-algebras $(C^+_{m,n})$, whose K-theory can be computed by a short exact sequence.

Definition 7.27 ([20, Proposition 8.7]). Define

$$C^+_{m,n} := AC^{Y+}_{m,n} \cap B^+_{\mathscr{B}}.$$

Then every $C^+_{m,n}$ $(m < n)$ is a C^*-subalgebra of $C^+_{\mathscr{B}}$.

Proposition 7.28 ([20, Theorem 8.13]). *Define $C^+_{m,n}$ as above. Then:*

1. *$C^+_{m,n} \subseteq C^+_{m-1,n+1}$. So $(C^+_{-n,n})$ is an inductive system.*
2. *$C^+_{\mathscr{B}}$ is the inductive limit C^*-algebra given by the inductive system, that is,*

$$C^+_{\mathscr{B}} = \overline{\bigcup_n C^+_{-n,n}}.$$

K-theory of $C^+_{m,n}$ can be computed by a short exact sequence, which uses another inductive system of groupoids $(H_{m,n})$, each $H_{m,n}$ being a subgroupoid of $G_{m,n}$. The construction requires the standing assumption. More details can be found in [20, Proposition 7.15, Page 57]. The definition there is neither well-organised nor clear enough, so I do not intend to provide a complete description of the groupoids $(H_{m,n})$ here.

Proposition 7.29 ([20, Corollary 8.14]). *There is a short exact sequence*

$$\bigoplus_{v \in V_n} A^+_{m,n,v} \otimes \mathrm{C}_0(0, \nu_s(v)) \rightarrowtail C^+_{m,n} \twoheadrightarrow \mathrm{C}^*(H_{m,n}).$$

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